# KAZHDAN-LAUMON REPRESENTATIONS OF CHEVALLEY GROUPS, CHARACTER SHEAVES AND SOME GENERALIZATION OF THE LEFSCHETZ-VERDIER TRACE FORMULA

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ABSTRACT. Let $G$ be a reductive group over a finite field $\mathbb{F}_q$ and let $G(\mathbb{F}_q)$ be its group of $\mathbb{F}_q$-rational points. In 1976 P. Deligne and G. Lusztig (following a suggestion of V. Drinfeld for $G = SL(2)$) associated to any maximal torus $T$ in $G$ defined over $\mathbb{F}_q$ certain remarkable series (virtual) representations of the group $G(\mathbb{F}_q)$. These representations were realized in é tale $\ell$-adic cohomology of certain algebraic varieties over $\mathbb{F}_q$, (called the Deligne-Lusztig varieties). Using these representations G. Lusztig has later given a complete classification of representations of $G(\mathbb{F}_q)$. He has also discovered that characters of these representations might be computed from certain geometrically defined $\ell$-adic perverse sheaves on the group $G$, called character sheaves.

Despite the fact that both Deligne-Lusztig representations and character sheaves are defined by some simple (and beautiful) algebro-geometric procedures, the proof of the relation between the two is rather complicated (apart from the case, when $T$ is split, where it becomes just the usual formula for the character of an induced representation). In 1987 D. Kazhdan and G. Laumon proposed a different (conjectural) geometric way for constructing representations of $G(\mathbb{F}_q)$. Their idea was based on exploiting the generalized Fourier-Deligne on the basic affine space $X = G/U$ of $G$ (here $U$ is a maximal unipotent subgroup of $G$). In the first part of this paper we give a rigorous definition of (almost all) Kazhdan-Laumon representations. Namely for every $\mathbb{F}_q$-rational maximal torus $T$ in $G$ and every quasi-regular (*non-singular* in the terminology of [6]) character $\theta : T(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^*$ we construct certain representation $V_\theta$ of $G(\mathbb{F}_q)$. We show that this representation is finite-dimensional and that it is irreducible if $\theta$ is regular (*in a generic position* in the terminology of [6]). Moreover, it follows from the proof of irreducibility in essentially tautological way, that the character of $V_\theta$ is given by the trace function of the corresponding Lusztig's character sheaf. Roughly speaking, all the above is achieved by replacing the generalized Fourier-Deligne transformations on $X$ by the suitable version of Radon transformations.

It follows from the above computation of the character of $V_\theta$, that $V_\theta$ is equivalent to the corresponding Deligne-Lusztig representation $R_\theta$. The second part of this paper is devoted to the construction of an explicit isomorphism between the two. This is done by using certain generalization of the Lefschetz-Verdier trace formula for Radon transformations, which we cannot establish in the full generality (however, we believe that this is a technical difficulty, which is there only because of authors' laziness).

## CONTENTS

## 1. Introduction

**1.1. Preliminaries.** Let $p$ be an prime number, $q$ – a power of $p$ and let $\mathbb{F}_q$ denote the finite field with $q$ elements. For any scheme $X$ defined over $\mathbb{F}_q$ we denote by $X(\mathbb{F}_q)$ the set of all $\mathbb{F}_q$ -rational points of of $X$. Let also $\mathrm{Fr}: X \to X$ denote the geometric Frobenius morphism. We will also fix a prime number $\ell \neq p$.

We will denote by $\mathcal{D}(X)$ the bounded derived category of $\ell$-adic constructible sheaves on $X$ and by $\mathrm{Perv}(X)$ the corresponding category of perverse sheaves.

**1.2. Deligne-Lusztig representations.** Let $G$ be an algebraic reductive connected group defined over $\mathbb{F}_q$ (in fact, for the sake of simplicity, we will speak mostly about semisimple simply connected groups; however, everything done in this paper goes through for any reductive group over $\mathbb{F}_q$ with only minor change)s. The representations of the finite group $G(\mathbb{F}_q)$ were classified by G. Lusztig. As his main tool, G. Lusztig used an earlier construction of some amount of (virtual) representations of $G(\mathbb{F}_q)$ by P. Deligne and G. Lusztig. Let us recall some facts about this construction.

Recall (cf. [6]) that one can associate to $G$ its abstract Cartan group $T$, together with canonical $\mathbb{F}_q$-rational structure on it and a root system (with the chosen system of positive roots). Let $W$ be the Weyl group of $G$ which will be considered as a subgroup of the group $\mathrm{Aut}(T)$ of automorphisms of $T$. Any $w \in W^{\mathrm{Fr}}$ defines a new $\mathbb{F}_q$- rational structure on $T$ by composing the old action of Fr with $w$. We will denote

this Frobenius morphism by $\mathrm{Fr}_w$. Let $T_w(\mathbb{F}_q)$ denote the group of $\mathbb{F}_q$-rational points of $T$ with the above $\mathbb{F}_q$-structure. Thus $T_w(\mathbb{F}_q) = T(\overline{\mathbb{F}}_q)^{\mathrm{Fr}_w}$.

For any $w \in W$ Deligne and Lusztig have defined certain algebraic variety $X_w$ over $\overline{\mathbb{F}}_q$ endowed with an action of the group $G(\mathbb{F}_q) \times T_w(\mathbb{F}_q)$. Thus we may consider the virtual representation $\sum(-1)^i H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$ of $G(\mathbb{F}_q)$ and decompose it with respect to characters of $T_w(\mathbb{F}_q)$. For any character $\theta:\ T_w(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^{\,*}$ we will denote by $R_{\theta,w}$ the corresponding virtual representation of $G(\mathbb{F}_q)$. In their paper Deligne and Lusztig showed the following

**Theorem 1.** *Suppose that $\theta$ is regular (cf.* 3.1*; note, that what we call regular character here is called a character in a generic position in* [6]*). Then $R_{\theta,w}$ is an irreducible representation of $G(\mathbb{F}_q)$ (in particular, it is a genuine representation, not a virtual one).*

**1.3. Character sheaves.** Let $\mathcal{L}$ be a one-dimensional tame ( cf. 3.1) $\ell$-adic local system on $T$. Suppose that we have also fixed an isomorphism $\mathcal{L} \simeq \mathrm{Fr}_w^*\mathcal{L}$ for some $w \in W$. In [14] G. Lusztig has defined certain $\mathrm{Ad}G$-equivariant perverse sheaf $\mathcal{K}_\mathcal{L}$ on the group $G$, equipped with the structure of a Weil sheaf over $\mathbb{F}_q$ (in the notations of [14] this is the sheaf $\mathcal{K}^e_\mathcal{L}$, where $e$ denotes the unit element in $W$). Let $\mathrm{tr}(\mathcal{K}_\mathcal{L})$ denote the "trace function" of $\mathcal{K}_\mathcal{L}$ (by the Grothendieck faicseaux-fonctions correspondence). This is an $\mathrm{Ad}G(\mathbb{F}_q)$-equivariant function on $G(\mathbb{F}_q)$ with values in $\overline{\mathbb{Q}}_\ell$. The following result is due to G. Lusztig (cf., for example, [17]).

**Theorem 2.** *One has*

$$\mathrm{tr}(\mathcal{K}_\mathcal{L}) = \chi(R_{\theta,w}) \tag{1.1}$$

*where $\chi(R_{\theta,w})$ is the character of the virtual representation $R_{\theta,w}$.*

**1.4. Kazhdan-Laumon representations.** Despite the fact that both sides of the equality (1.1) are defined by simple geometric procedures, the proof of (1.1) is not as straightforward as one would expect it (except for the case $w = 1$ where it becomes just the usual formula for the character of an induced representation). This phenomenon might have 2 explanations.

1) The Deligne-Lusztig varieties $X_w$ change as we extend the base field $\mathbb{F}_q$, which makes difficult the analysis of the behaviour of $R_{\theta,w}$ under extensions of the base field.

2) The Weil structure on the sheaf $\mathcal{K}_{\mathcal{L}}$ is defined in a rather inexplicit way (it is defined explicitly on the set $G_{rs}$ of regular semisimple elements in $G$, and then one defines it on the whole of $G$ by making use of the fact that $\mathcal{K}_{\mathcal{L}}$ is an intermediate extension of its restriction to $G_{rs}$).

Let now $\mathcal{L}$ be a tame quasi-regular one-dimensional local system on $T$ (this means that $\alpha^*\mathcal{L}$ is not isomorphic to the constant sheaf for any coroot $\alpha : \mathbb{G}_m \to T$) endowed with an isomorphism $\mathcal{L} \simeq \mathrm{Fr}_w^*\mathcal{L}$. One of the purposes of this paper is to give a simple geometric proof of the fact that $\mathrm{tr}(\mathcal{K}_{\mathcal{L}})$ is a character of some irreducible representation of $G(\mathbb{F}_q)$. To do that we use a different construction of representations of $G(\mathbb{F}_q)$ suggested by D. Kazhdan and G. Laumon in [11]. We also give a different way to define the structure of a Weil sheaf on $\mathcal{K}_{\mathcal{L}}$. Both these steps are done by exploiting the generalized Fourier-Deligne transformations on the basic affine space $X = G/U$ ($U$ is a maximal unipotent subgroup of $G$).

Let us give a brief recollection of the variant of the Kazhdan-Laumon construction which will be used in this paper. We would like to imitate "as much as possible" the construction of principle series. Recall that the latter are defined as follows. Let $X$ be again the basic affine space of $G$. This is a quasi-affine algebraic variety defined over $\mathbb{F}_q$. It admits a natural action of the group $G \times T$ ($T$ acts there since we may choose a maximal torus in $G$ which normalizes $U$). Therefore, the set $X(\mathbb{F}_q)$ admits a natural action of $G(\mathbb{F}_q) \times T(\mathbb{F}_q)$. Hence, given a character $\theta$ of $T(\mathbb{F}_q)$ we may construct a representation $V(\theta)$ of $G(\mathbb{F}_q)$ (this is just the representation of $G(\mathbb{F}_q)$ in the space of functions on $X(\mathbb{F}_q)$ which change according to character $\theta$ under the action of $T(\mathbb{F}_q)$).

Now, we would like to construct series of representations which are parametrised by characters of other maximal tori in $G$ which are defined over $\mathbb{F}_q$. According to [6] different conjugacy classes of maximal tori in $G$, which are defined over $\mathbb{F}_q$ are parametrised by twisted conjugacy classes in $W$ (if $G$ is split, then twisted conjugacy classes are just the same as the usual conjugacy classes; in the general case cf. 3.1 for the definition).

In [11] Kazhdan and Laumon have described certain idea how this can be done. Unfortunately, their construction relied on some conjectures from homological algebra which are still unknown. However, one can use certain modification of the Kazhdan-Laumon construction, which is already well-defined.

It is observed in [11] that the category $\mathrm{Perv}(X)$ of $\ell$-adic perverse sheaves on $X$ admits a natural action of the braid group, corresponding to $W$ (cf. 2.3). In 2.3 we define certain subcategory $\mathrm{Perv}^0(X)$ of $\mathrm{Perv}(X)$, on which the above action of the braid group can be reduced to an action of $W$. This subcategory is "big enough" (in particular, it is still invariant under the action of $G \times T$).

Consider the category $\mathrm{Perv}^0_w$ of "$w$-Weil sheaves" on $X$. An object of $\mathrm{Perv}^0_w$ is an object $A$ of $\mathrm{Perv}^0(X)$ together with an isomorphism $\alpha : A \simeq \mathrm{Fr}^*\Phi_w(A)$ where $\Phi_w$ is the functor on $\mathrm{Perv}^0(X)$ defining the action of $W$ on it and $\mathrm{Fr} : X \to X$ is the corresponding Frobenius morphism. For $w = 1$ we get just the usual notion of Weil sheaves. Consider now the space $K := K(\mathrm{Perv}^0(X)) \otimes \overline{\mathbb{Q}}_\ell$ ($K(\mathrm{Perv}^0(X))$ is the Grothendieck group of $\mathrm{Perv}^0(X)$). This space is an infinite-dimensional representation of the group $G(\mathbb{F}_q) \times T_w(\mathbb{F}_q)$. Next, one can define certain quotient $V_w$ of this space. The quotient is defined in such a way that for $w = 1$ if we replace $\mathrm{Perv}^0(X)$ by $\mathrm{Perv}(X)$ then we get just the space of functions on the finite set $X(\mathbb{F}_q)$. For a character $\theta$ of $T_w(\mathbb{F}_q)$ we denote $V_{\theta,w}$ the corresponding representation of $G(\mathbb{F}_q)$.

**Theorem 3.** *Let $\theta$ be any character of $T_w(\mathbb{F}_q)$. Then the Kazhdan-Laumon representation $V_{\theta,w}$ is finite-dimensional and nonzero. If $\theta$ is regular ( in a generic position in the terminology of* [6]*) then $V_{\theta,w}$ is irreducible. Moreover, if $\theta$ is non-singular (in the sense of* [6]*) then the character of $V_{\theta,w}$ is equal to $\mathrm{tr}(\mathcal{K}_{\mathcal{L}})$, where $\mathcal{L}$ is the one-dimensional local system on $T$, which corresponds to $\theta$. The last fact is established by a simple geometric argument which is essentially "not different" from the case of principal series).*

One can also write down an explicit geometric isomorphism between $V_{\theta,w}$ and the corresponding Deligne-Lusztig representation (this is done in Section 7). This construction requires certain generalization of the Lefschetz-Verdier trace formula, which is the subject of Section 6. In this way we obtain a new proof of the fact that characters of the Deligne-Lusztig representations can be computed using the

corresponding character sheaves (modulo the assumption that our generalized trace fromula holds in the case we need).

**1.5. Contents.** This text is organized as follows. In section 2 we establish some facts about the Fourier-Deligne transforms on the basic affine space, which will be needed in the sequel. In Section 3 we recall some basic facts about the sheaf $\mathcal{K}_{\mathcal{L}}$ and give a definition of the Weil structure on $\mathcal{K}_{\mathcal{L}}$ which is slightly different from that of G. Lusztig and uses the Fourier-Deligne transforms on $X$. In Section 4 we give the precise definition of Kazhdan-Laumon representations and state our main results about them. Section 5 is devote to the proof of these results. Finally, in Section 6 we discuss certain generalization of the Lefschetz-Verdier trace formula and in Section 7 use this generalization in order to construct a geometric isomorphism between Kazhdan-Laumon and Deligne-Lusztig representations.

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## 2. Generalized Fourier-Deligne and Radon transformations on the basic affine space

### 2.1. Fourier-Deligne and Radon transformations on a vector bundle.

2.1.1. *Definition of Fourier transform.* Let $S$ be a scheme of finite type over $k=\overline{\mathbb{F}}_q$ and let $\pi_E : E \to S$ be a vector bundle over $S$ of rank $r$ and let $\pi_{E^\vee} : E^\vee \to S$ be the dual bundle. Fix a nontrivial character $\psi$ of $\mathbb{F}_q$ and denote by $\mathbf{Four}_\psi : \mathcal{D}(E) \to \mathcal{D}(E^\vee)$ the Fourier-Deligne transform (cf. [9]). Recall that it is defined as follows. Let $\mathcal{L}_\psi$ denote the Artin-Schreier sheaf on $\mathbb{A}^1 = \mathbb{A}^1_k$. Let also $\mu :\ E \times_S E^\vee \to \mathbb{A}^1$ denote the pairing map and let $\mathrm{pr}_1, \mathrm{pr}_2$ denote the projections from $E \times_S E^\vee$ to the

first and to the second variable respectively. Then, by definition

$$\mathbf{Four}_\psi(A) = \mathrm{pr}_{2!}(\mathrm{pr}_1^* A \otimes \mu^* \mathcal{L}_\psi)[r] \tag{2.1}$$

for any $A \in \mathcal{D}(E)$. We will omit the subscript "$\psi$" when it does not lead to a confusion.

Let $inv:\ \mathbb{G}_m \to \mathbb{G}_m$ denote the inverseion map (i.e. $inv(\lambda) = \lambda^{-1}$ and let $m_E:\ \mathbb{G}_m \times E \to E$ denote the multiplication map (defined by $m_E(\lambda, e) = \lambda e$). Then it is easy to see that one has a canonical isomorphism of functors

$$(\mathrm{id}\times\mathbf{Four}) \circ m_E^* \simeq (inv \times \mathrm{id})^* \circ m_{E^\vee}^* \circ \mathbf{Four} \tag{2.2}$$

2.1.2. *Definition of Radon transform.* Let us now define the version of the Radon transformation that will be needed later. The notations will be as above. For a vector bundle $\pi_E : E \to X$ let us denote by $j_E : \widetilde{E} \to E$ the embedding of the complement to the zero section on $E$. Let now $Z_E \subset E \times E^\vee$ be the closed subvariety in $E \times E^\vee$ defined by

$$Z_E = \{(e, e^\vee) \in E \times E^\vee |\ \langle e, e^\vee \rangle = 1\} \tag{2.3}$$

Note that $Z$ lies, in fact, inside $\widetilde{E} \times \widetilde{E}^\vee$. Let $p_1, p_2$ denote respectively the projections from $Z$ to $\widetilde{E}$ and $\widetilde{E}^\vee$. We now define the Radon transform $\mathbf{Rad}:\ \mathcal{D}(\widetilde{E}) \to \mathcal{D}(\widetilde{E}^\vee)$ by

$$\mathbf{Rad}(A) = p_{2!} p_1^*(E)[r-1] \tag{2.4}$$

2.1.3. *The categories $\mathcal{D}^{mon}(E)$, $\mathcal{D}^0(E)$ and $\mathcal{D}^{reg}(E)$.* The main difference between the Fourier-Deligne transform and the Radon transform, is that the latter does not map in general perverse sheaves into perverse sheaves. However, below we will define certain full subcategory of $\mathcal{D}(E)$ (resp. $\mathcal{D}(E^\vee)$), which we will denote by $\mathcal{D}^0(E)$ (resp. $\mathcal{D}^0(E^\vee)$), such that the restrictions of the functors **Four** and **Rad** to $\mathcal{D}^0(E)$ will be canonically isomorphic. This, in particular, will imply that **Rad** maps perverse objects, lying in $\mathcal{D}^0(E)$, into perverse ones.

First of all, let us define the category of monodromic sheaves.

**Definition 1.** *1) A complex $A \in \mathcal{D}(E)$ is called monodromic if there exists $n \in \mathbb{N}$ such that for every $i \in \mathbb{Z}$, $H^i(A)$ has a filtration, whose factors are equivariant with respect to the action of the group $\mathbb{G}_m$ on $E$, defined by $\lambda(e) = \lambda^n e$ for $\lambda \in \mathbb{G}_m$*

*and $e \in E$ (here $H^i(A)$ denotes the $i$-th perverse cohomology of $A$). We denote by $\mathcal{D}^{\mathrm{mon}}(E)$ the full subcategory of $\mathcal{D}(E)$ consisting of monodromic complexes.*

*2) A monodromic complex $A$ is called regular if for every $i \in \mathbb{Z}$, $H^i(A)$ has a filtration, whose quotients are $(\mathbb{G}_m, \mathcal{L})$-equivariant, where $\mathcal{L}$ is some non-constant local system on $\mathbb{G}_m$. We denote by $\mathcal{D}^{\mathrm{reg}}(E)$ the full subcategory of $\mathcal{D}(E)$, consisting of regular monodromic complexes.*

*Remark* 1. The word "equivariant" in Definition 1 means, in fact, "can be given an equivariant structure".

*Remark* 2. Part 1 of Definition 1 makes sense for sheaves on arbitrary variety $X$, endowed with an action of an algebraic torus $T$.

*Remark* 3. Let $X$ be any variety and let $A \in \mathcal{D}(X)$. Supose that for every $i \in \mathbb{Z}$ we are given a filtration of $H^i(A)$ with certain quotients. Then we will say that $A$ *is glued* from those quotients.

We will define now a third category $\mathcal{D}^0(E)$, which will be intermediate between $\mathcal{D}^{\mathrm{reg}}(E)$ and $\mathcal{D}^{\mathrm{mon}}(E)$ (i.e. we will have natural inclusions $\mathcal{D}^{\mathrm{reg}}(E) \subset \mathcal{D}^0(E) \subset \mathcal{D}^{\mathrm{mon}}(E)$ of full subcategories).

Let $i_E : X \to E$ denote the embedding of the zero section. By definition, the category $\mathcal{D}^0(E)$ is a full subcategory of $\mathcal{D}^{\mathrm{mon}}(E)$, consisting of objects $A$ of $\mathcal{D}^{\mathrm{mon}}(E)$, such that

1) $\pi_{E*}A = \pi_{E!}A = 0$
2) $i_E^*A = i_E^!A = 0$

Note, that it follows from the definition that $\mathcal{D}^0(E)$ can be identified with the full subcategory of $\mathcal{D}^{\mathrm{mon}}(\widetilde{E})$, consisting of those $A \in \mathcal{D}^{\mathrm{mon}}(\widetilde{E})$ for which the canonical morphism $j_{E!}A \to j_{E*}A$ is an isomorphism and $\pi_{E*}(j_{E*}A) = \pi_{E!}(j_{E!}A) = 0$. Therefore, we will sometimes write $\mathcal{D}^0(\widetilde{E})$ instead of $\mathcal{D}^0(E)$, in order to emphasize, that the objects of this category may be regarded as sheaves on $\widetilde{E}$.

We will denote by $\mathrm{Perv}^{\mathrm{mon}}(E)$, $\mathrm{Perv}^0(E)$ and $\mathrm{Perv}^{\mathrm{reg}}(E)$ the corresponding categories of perverse sheaves.

**Theorem 4.** *1) Both functors* ***Rad*** *and* ***Four*** *map $\mathcal{D}^{\mathrm{mon}}(E)$ to $\mathcal{D}^{\mathrm{mon}}(E^\vee)$, $\mathcal{D}^{\mathrm{reg}}(E)$ to $\mathcal{D}^{\mathrm{reg}}(E^\vee)$ and $\mathcal{D}^0(E)$ to $\mathcal{D}^0(E^\vee)$ (more precisely, one should say that* ***Rad*** *maps $\mathcal{D}^{\mathrm{mon}}(\widetilde{E})$ to $\mathcal{D}^{\mathrm{mon}}(\widetilde{E}^\vee)$).*

*2) One has canonical isomorphism*

$$\boldsymbol{Rad} \simeq j_E^* \circ \boldsymbol{Four} \circ j_{E!} \tag{2.5}$$

*of functors, going from $\mathcal{D}^{mon}(\widetilde{E})$ to $\mathcal{D}^{mon}(\widetilde{E}^\vee)$.*

*Proof.* The proof of point (1) of Theorem 4 is completely straightforward and it is left to the reader. The proof of (2) is essentially a repetition of the proof of Theorem 9.13 in [3]. Let us, however, present it for the sake of completeness.

The proof follows from the following result.

**Lemma 1.** *Let $Y$ be a scheme over $\overline{\mathbb{F}}_q$. Let $K \in \mathcal{D}^{mon}(\mathbb{G}_m \times Y)$ and let $\tau : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ denote the natural embedding. Then one has canonical isomorphism*

$$pr_!((\tau \times \mathrm{id})_! K \otimes \mathcal{L}_\psi) \simeq i_1^* K[-1] \tag{2.6}$$

*where $i_1$ denotes the embedding of $1 \times Y$ in $\mathbb{G}_m \times Y$.*

The proof is left to the reader.

Let us now show how the lemma implies point (2) of the theorem. Consider the following diagram:

$$\begin{tikzcd}
 & & \widetilde{E} \underset{S}{\times} E^\vee \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "j \times \mathrm{id}"] & & \\
 & \widetilde{E} \arrow[dr, "j"'] & & E \underset{S}{\times} E^\vee \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"] & \\
 & & E & & E^\vee
\end{tikzcd}$$

Let $\widetilde{\mathrm{pr}}_2$ denote the composition of $\mathrm{pr}_2$ with $j \times \mathrm{id}$. Then it follows that

$$\mathbf{Four}(j_! A) \simeq \widetilde{\mathrm{pr}}_{2!}(\mathrm{pr}_1^* A \otimes (j \times \mathrm{id})^* \mu^* \mathcal{L}_\psi)[r] \tag{2.7}$$

On the other hand, $\widetilde{\mathrm{pr}}_2 = \mathrm{pr} \circ (\mu, \mathrm{pr}_2)$, where $\mu : \widetilde{E} \times_S E^\vee \to \mathbb{A}^1$ is the pairing map and $\mathrm{pr} : \mathbb{A}^1 \times E^\vee$ is the projection to the second multiple. Then

$$\mathbf{Four}(j_! A) \simeq (\mathrm{pr} \circ (\mu, \mathrm{pr}_2))_! (\mathrm{pr}_1^* A \otimes (j \times \mathrm{id})^* \mu^* \mathcal{L}_\psi)[r] \simeq \tag{2.8}$$

$$\mathrm{pr}_! ((\mu, \mathrm{pr}_2)_! \mathrm{pr}_1^* A \otimes (\mathcal{L}_\psi \boxtimes \overline{\mathbb{Q}}_{l,E^\vee}))[r] \simeq i_1^* (\mu, \mathrm{pr}_2)_! \mathrm{pr}_1^* A[r-1] \tag{2.9}$$

where the last two isomorphisms are given respectively by the projection formula and Lemma 1. On the other hand (by base change) $i_1^*(\mu \times \mathrm{id})_! \mathrm{pr}_1^* A[r-1]$ is isomorphic to $p_{2!} p_1^* A[r-1]$, which finishes the proof. $\square$

**Corollary 1.** *Let $A \in \mathrm{Perv}^{mon}(\widetilde{E})$ and suppose that $j_{E!}A$ is perverse. Then $\boldsymbol{Rad}(A)$ is perverse. In particular, $\boldsymbol{Rad}$ maps $\mathrm{Perv}^0(E)$ to $\mathrm{Perv}^0(E)$.*

**Warning 1.** It follows from the definition, that both functors **Four** and **Rad** "commute" with the functor $\mathrm{Fr}^*$ of inverse image with respect to the Frobenius morphism. This means that we have natural isomorphisms of functors $\nu_F : \mathbf{Four} \circ \mathrm{Fr}^* \widetilde{\to} \mathrm{Fr}^* \circ \mathbf{Four}$ and $\nu_R : \mathbf{Rad} \circ \mathrm{Fr}^* \widetilde{\to} \circ \mathrm{Fr}^* \mathbf{Rad}$. During the proof of Theorem 4 we have constructed an isomorphism between the restrictions of **Rad** and **Four** on the category $\mathcal{D}^0(E)$. However, this isomorphism does not transform $\nu_F$ into $\nu_R$ (even if we change $\nu_F$ or $\nu_R$ by a Tate twist). This is the reason why Radon transform (in the above sense) is not equal to Fourier transform, when one passes from sheaves to functions on $E(\mathbb{F}_q)$ (in fact, the difference between Radon and Fourier transform on the level of functions is given by certain $\Gamma$-function, attached to the field $\mathbb{F}_q$).

**Warning 2.** The term "Radon transform" is usually used in the literature for a little different transformation, going from $\mathcal{D}(\mathbb{P}(E))$ to $\mathcal{D}(\mathbb{P}(E^\vee))$, where $\mathbb{P}(E)$ and $\mathbb{P}(E^\vee)$ denote the corresponding projectivized bundles (cf. [3], for example). We would like to note that if one identifies $\mathcal{D}(\mathbb{P}(E))$ with the derived category of $\mathbb{G}_m$-equivariant sheaves on $\widetilde{E}$, then our Radon transformation does not coincide with that of [3]. For example, point 2 of Theorem 4 for the Radon transform discussed in [3] does not hold in general, without some additional hypopaper on $A$ (it does hold, however, for $\mathbb{G}_m$-equivariant $A \in \mathcal{D}^0(E)$).

**2.2. Fourier transform on a symplectic vector bundle.** Suppose now that our vector bundle $\pi_E : E \to S$ is endowed with some fiberwise symplectic structure $\omega$. Then, using $\omega$ we can identify $E$ with $E^\vee$ and hence, we may consider Fourier

transform as a functor from $\mathcal{D}(E)$ to itself. Let us denote this functor simply by $\mathcal{F}(n)$. I.e. $\mathcal{F}$ is the symplectic Fourier-Deligne transform with a Tate twist by $-n$ (of course, one should remember that $\mathcal{F}$ depends also on $\psi$). The following result is due to A. Polishchuk ( cf. [19]). This result will not be used in the sequel, but it is useful to keep it in mind.

**Proposition 1.** *There exists canonical isomorphism of functors $c: \ \mathcal{F}^2 \tilde{\to} Id$. The pair $(\mathcal{F}(n), c)$ defines an action of the group $\mathbb{Z}_2$ on the category $\mathcal{D}(E)$ (cf.* [5] *for this notion), i.e. $c$ satisfies the following associativity condition:*

(2.10) *the two morphisms $\mathcal{F} \circ c$ and $c \circ \mathcal{F}$ (from $\mathcal{F}^3(3n)$ to $\mathcal{F}(n)$) coincide*

**Corollary 2.** *The categories $\mathcal{D}^{mon}(E)$ and $\mathcal{D}^0(E) = \mathcal{D}^0(\widetilde{E})$ admit a natural action of the group $\mathbb{Z}_2$ (given by the symplectic Fourier-Deligne transform).*

### 2.3. Fourier-Deligne transforms on the basic affine space.

2.3.1. *Action of a monoid on a category.* Let $M$ be a monoid and let $\mathcal{C}$ be a category.

**Definition 2.** *An action of $M$ on $\mathcal{C}$ consists of the following data (* cf. [5]*):*

- *a functor $F_m: \ \mathcal{C} \to \mathcal{C}$ for every $m \in M$*
- *an isomorphism of functors*

$$\alpha_{m_1,m_2}: \ F_{m_1} \circ F_{m_2} \to F_{m_1 m_2}$$

*for every $m_1, m_2 \in M$ such that $m_1 m_2$ is defined.*

*This data should satisfy the following associativity condition:*
*For every $m_1, m_2, m_3 \in M$, such that $m_1 m_2 m_3$ is defined one has*

$$\alpha_{m_1 m_2} \circ \alpha_{m_1,m_2} = \alpha_{m_1,m_2 m_3} \circ \alpha_{m_2,m_3} \tag{2.11}$$

*(both sides are isomorphisms between $F_{m_1} \circ F_{m_2} \circ F_{m_3}$ and $F_{m_1 m_2 m_3}$.*

We will be interested in the particular case of a *braid monoid* acting on various categories. By definition, a braid monoid associated to a reductive group $G$ is the semigroup with generators $s_\alpha$ for every simple root $\alpha$ and with braid relations as the only relations. Braid monoid actions on categories were studied in [5].

2.3.2. *The basic affine space of* $G$. From now until the end of this section we assume that $G$ is semisimple and simply connected. Let $\mathcal{B}$ denote the flag variety of $G$. By definition this is the variety of all Borel subgroups in $G$.

We now want to define certain $T$-torsor $\eta : X \to \mathcal{B}$. Unlike the flag variety, the basic affine space $X$ cannot be attached to the group $G$ canonically. In order to define it we have to fix the following data. Recall that $T$ denotes the abstract Cartan group of $G$. Let $\omega_1, ...\omega_n$ denote the fundamental weights of $G$. Then we must fix the corresponding line bundles $\mathcal{O}(\omega_1), ..., \mathcal{O}(\omega_n)$ on $\mathcal{B}$, which are *a priori* defined uniquely up to a non-canonical isomorphism (fixing of $\mathcal{O}(\omega_i)$ is the same as fixing of the corresponding representation of $G$ with highest weight $\omega_i$, which is also defined uniquely up to a non-canonical isomorphism). By abuse of notation, we will denote by $\mathcal{O}(-\omega_i)$ also the total space of the line bundle $\mathcal{O}(-\omega_i)$. Then we define $X$ to be the complement to "all zero sections" in $\mathcal{O}(-\omega_1) \times_{\mathcal{B}} \times ... \times_{\mathcal{B}} \mathcal{O}(-\omega_n)$. Namely, let $\widetilde{\mathcal{O}}(-\omega_i)$ denote the complement to the zero section in $\mathcal{O}(-\omega_i)$. Then we define

$$X = \widetilde{\mathcal{O}}(-\omega_1) \underset{\mathcal{B}}{\times} ... \underset{\mathcal{B}}{\times} \widetilde{\mathcal{O}}(-\omega_n) \tag{2.12}$$

It is easy to see that

$$\mathcal{A} = \Gamma(X, \mathcal{O}_X) = \bigoplus_{\lambda \in P^+(G)} \Gamma(\mathcal{B}, \mathcal{O}(\lambda)) \tag{2.13}$$

with the obvious multiplication there (here $P^+(G)$ denotes the set of integral dominant weights of $G$). The variety $X$ is an open subset of the affine variety $\overline{X} = \operatorname{Spec} \mathcal{A}$. The complement $\overline{X} \backslash X$ is defined by the ideal

$$J = \bigoplus_{\lambda \in P^{++}(G)} \Gamma(\mathcal{B}, \mathcal{O}(\lambda)) \tag{2.14}$$

where $P^{++}(G)$ denotes the set of integral dominant regular weights of $G$.

It is easy to see, that if we choose a Borel subgroup $B \subset G$ with unipotent radical $U$, then $X$ can be identified with $G/U$. We will often make such a choice in order to simplify the discussion.

2.3.3. *Fourier transforms on the basic affine space.* For a simple root $\alpha$ let $P_\alpha \subset G$ be the minimal parabolic of type $\alpha$ containing $B$. Let $B_\alpha$ be the commutator subgroup

of $P_\alpha$, and denote $X_\alpha := G/B_\alpha$. We have an obvious projection of homogeneous spaces $\pi_\alpha : X \to X_\alpha$. It is a fibration with the fiber $B_\alpha/U = \mathbb{A}^2 - \{0\}$.

Let $\overline{\pi}_\alpha : \overline{X}^\alpha \to X_\alpha$ be the relative affine completion of the morphism $\pi_\alpha$. (So $\overline{\pi}_\alpha$ is the affine morphism corresponding to the sheaf of algebras $\pi_{\alpha*}(\mathcal{O}_X)$ on $X_\alpha$.) Then $\overline{\pi}_\alpha$ has the structure of a 2-dimensional vector bundle; $X$ is identified with the complement to the zero-section in $\overline{X}^\alpha$. The $G$-action on $X$ obviously extends to $\overline{X}^\alpha$; moreover, it is easy to see that the determinant of the vector bundle $\overline{\pi}_\alpha$ admits a canonical (up to a constant) $G$-invariant trivialization, i.e. $\overline{\pi}_\alpha$ admits unique up to a constant $G$-invariant fiberwise symplectic form $\omega_\alpha$.

**Examples.** 1) Let $G = SL(2)$ and let $\alpha$ denote the unique simple root of $G$. Then $X \simeq \mathbb{A}^2\backslash\{0\}$, $X_\alpha = pt$ and $X^\alpha \simeq \mathbb{A}^2$, i.e. $X^\alpha$ is isomorphic to the defining representation of $SL(2)$. It is clear that this vector space admits unique up to a constant symplectic form, which is automatically $G$-invariant.

2) Let $G = SL(n)$. Let $V$ denote the defining representation of $G$, i.e. $V$ is an $n$-dimensional vector space over $\overline{\mathbb{F}}_q$ with the natural $SL(n)$-action. Let us choose some identification of $\Lambda^n V$ with $\overline{\mathbb{F}}_q$, i.e. choose a nonzero element $\varepsilon \in \Lambda^n V$. Then the basic affine space of $G$ may be described as follows.

(2.15)
$$X = \{(0 \subset V_1 \subset V_2 \subset ... \subset V_n = V, \varepsilon_i \in V_i/V_{i-1})| \quad \text{such that } \varepsilon_1 \wedge \varepsilon_2 \wedge ... \wedge \varepsilon_n = \varepsilon\}$$

Of course, $\varepsilon_n$ is uniquely determined from $\varepsilon_1, ..., \varepsilon_{n-1}$ (so we could omit the $\varepsilon_n$ in the definition, adding the condition $\varepsilon_i \neq 0$ for $i = 1, ..., n-1$).

Let $\alpha_k$ denote denote the $k$-th simple root of $SL(n)$ $(k = 1, ..., n-1)$. Then

$$X_{\alpha_k} = \{(0 \subset V_1 \subset ... \subset V_{k-1} \subset V_{k+1} \subset ... \subset V_n = V, \\ 0 \neq \varepsilon_i \in V_i/V_{i-1}\| \text{ for } i \neq k, k+1, \text{ and } 1 \neq i \neq n\}$$

where $\dim V_i = i$. If we are given some point of $X_{\alpha_k}$ as above, then the fiber of $|pi_{\alpha_k}$ over this point is $V_{k+1}/V_{k-1} - \{0\}$ (since fixing of $\varepsilon_k$ obviously fixes $\varepsilon_{k+1}$) and the fiber of $\overline{\pi}_{\alpha_k}$ is $V_{k+1}/V_{k-1}$. Note that we have canonical element $\omega \in \Lambda^2(V_{k+1}/V_{k-1})$ which is defined by the condition

$$\varepsilon_1 \wedge ... \wedge \varepsilon_{k-1} \wedge \omega \wedge \varepsilon_{k+2} \wedge ...\varepsilon_n = \varepsilon$$

Hence we see that every fiber of $\overline{\pi}_{\alpha_k}$ is a 2-dimensional vector space with a symplectic structure, which is canonical once we choose $\varepsilon$.

Let us go back now to an arbitrary group $G$. Let $\alpha$ be a simple root of $G$. Then we may define a functor $\mathcal{F}_\alpha:\ \mathcal{D}(X)\to\mathcal{D}(X)$ by

$$\mathcal{F}_\alpha = j_\alpha^*\circ\mathcal{F}\circ j_{\alpha!} \tag{2.16}$$

where $j_\alpha:\ X\to\overline{X}^\alpha$ is the natural embedding and $\mathcal{F}$ is the Fourier-Deligne transform associated with the symplectic vector bundle $\overline{\pi}_\alpha:\ \overline{X}^\alpha\to X_\alpha$.

2.3.4. *The categories $\mathcal{D}^{mon}(X)$ and $\mathcal{D}^{reg}(X)$.* The category $\mathcal{D}^{\mathrm{mon}}(X)$ is defined analogously to $\mathcal{D}^{\mathrm{mon}}(E)$ (cf. Remark 2 after Definition 1). Note, that "monodromic" here means monodromic with respect to the action of $T$ on $X$.

**Definition 3.** *1) A tame local system $\mathcal{L}$ on $T$ is called quasi-regular if $\alpha^*\mathcal{L}$ is not constant for any coroot $\alpha:\ \mathbb{G}_m\to T$.*

*2) An object $A\in\mathcal{D}^{mon}(X)$ is called regular if it is glued from $(T,\mathcal{L})$-equivariant complexes, where $\mathcal{L}$ is a quasi-regular local system on $T$.*

The following result is proved in the next subsection.

**Theorem 5.** *The functors $\mathcal{F}_{s_\alpha}$ map $\mathcal{D}^{mon}(X)$ to $\mathcal{D}^{mon}(X)$. Moreover, there is a canonical action of the braid monoid $Br_W$, corresponding to $W$ on the category $\mathcal{D}^{mon}(X)$ which extends the functors $\mathcal{F}_{s_\alpha}$.*

We will now define a category $\mathcal{D}^0(X)$ (which is analogous to the category $\mathcal{D}^0(E)$) assuming Theorem 5. Namely, it follows from Theorem 5 that (since $W$ is canonically embedded into its braid monoid as a set) for any $w\in W$ we have canonical functor $\mathcal{F}_w:\ \mathcal{D}^{\mathrm{mon}}(X)\to\mathcal{D}^{\mathrm{mon}}(X)$. This functor maybe constructed as follows. Choose a reduced decomposition $w=s_{\alpha_1}...s_{\alpha_k}$ of $w$. Then $\mathcal{F}_w=\mathcal{F}_{s_{\alpha_1}}\circ...\circ\mathcal{F}_{s_{\alpha_k}}$. It follows from Theorem 5 that $\mathcal{F}_w$ does not depend on the choice of a reduced decomposition of $w$ up to a canonical isomorphism. We can now define the category $\mathcal{D}^0(X)$.

**Definition 4.** *The category $\mathcal{D}^0(X)$ is the full subcategory of $\mathcal{D}^{mon}(X)$, consisting of all the complexes $A\in\mathcal{D}^0(X)$ such that the following condition holds:*

*for any simple root $\alpha$ of $G$ and for any $w\in W$ the complex $\mathcal{F}_w(A)$ lies in $\mathcal{D}^0(\overline{X}^\alpha)$ (recall that $\overline{\pi}_\alpha:\ \overline{X}^\alpha\to X_\alpha$ is a 2-dimensional vector bundle).*

It is easy to see that one has canonical embeddings $\mathcal{D}^{\mathrm{reg}}(X) \subset \mathcal{D}^0(X) \subset \mathcal{D}^{\mathrm{mon}}(X)$ of full subcategories. We will also denote by $\mathrm{Perv}^{\mathrm{reg}}(X)$, $\mathrm{Perv}^0(X)$ and $\mathrm{Perv}^{\mathrm{mon}}(X)$ the corresponding categories of perverse sheaves.

One can show (cf. [19]) that the restrictions of the functors $\mathcal{F}_w$ to the category $\mathcal{D}^0(X)$ extend canonically to an action of $W$ on this category. However, this result is not an immediate corollary of the above and we will not present the proof in this paper, since we will not need this in the sequel.

## 2.4. Radon transforms on the basic affine space.

2.4.1. *The space $Z$.* Suppose again that we have fixed a basic affine space $X$ together with the symplectic forms $\omega_\alpha$ (cf. 2.3.3). Then we claim that there exists a natural $G$-torsor $Z$ (i.e. principal homogeneous space over $G$), endowed with a map $p: Z \to X$, an action of $T$ and an action of the braid monoid $\mathrm{Br}_W$, corresponding to $W$, such that the following conditions hold:

- $p$ is $G \times T$-invariant
- the actions of $\mathrm{Br}_W$ and $T$ on $Z$ agree with the action of $W$ on $T$
- the actions of $\mathrm{Br}_W$ and $G$ on $Z$ commute

First of all let $Z_{\mathcal{B}}$ be the space of all embeddings of $T$ into $G$. Clearly, this space admits a natural action of $G \times W$ (where $G$ acts by conjugation on itself and $W$ acts on $T$). Moreover, it is obvious that the $G$-action is transitive (in fact, a choice of a point $a : T \hookrightarrow G$ identifies $Z_{\mathcal{B}}$ with $G/\operatorname{Im} a$). On the other hand, we have a natural map $p_{\mathcal{B}} : Z_{\mathcal{B}} \to \mathcal{B}$, defined as follows. Let $a : T \hookrightarrow G$ be an embedding. Then there exists a unique Borel subgroup $B$ of $G$, containing the image of $T$, such that the pair $(T, a : T \hookrightarrow B)$ induces the canonical system of positive roots on $T$ (i.e. $B$ is characterized by the property that the set of roots of $T$ on the unipotent radical of the Lie algebra of $B$ is precisely the set of positive roots $R^+$ which the abstract Cartan group possesses). We now set $p_{\mathcal{B}}(a) = B$.

Now we define $Z = Z_{\mathcal{B}} \times_{\mathcal{B}} X$. Then we claim that $Z$ admits a natural action of $G \times \mathrm{Br}_W$ and an action of $T$. The actions of $G$ and $T$ on $Z$ are clear: $G$ acts on $Z = Z_{\mathcal{B}} \times_{\mathcal{B}} X$ diagonally (since $G$ acts both on $Z_{\mathcal{B}}$ and on $X$ this notion makes sense) and $T$ acts just on the second multiple. The action of $\mathrm{Br}_W$ on $Z$ may be described for example as follows.

First of all, let us describe this action for $G = SL(2)$. In this case

$$Z = \{x, y \in \mathbb{A}^2 |\ \omega(x, y) = 1\} \tag{2.17}$$

and we let the unique non-trivial element $s$ of the Weyl group of $SL(2)$ act by

$$s((x, y)) = (-y, x) \tag{2.18}$$

(note that $s$ has order 4).

For general $G$ we may now do the following. Assume that we have fixed the fundamental representations $V(\omega_i)$ of $G$. Then an element $z$ of $Z$ is the same as pair $T \subset B \subset G$ consising of a Borel subgroup $B$ together with a Cartan subgroup $T$ which is contained in $B$ plus a non-zero vector $v_i \in V(\omega_i)$ which is a highest weight vector of $G$ with respect to $B$ for every $i = 1, ..., \operatorname{rank}(G)$. Let now $\alpha$ be a simple root of $G$. We would like to define $s_\alpha(z)$ (here $s_\alpha \in W$ is the corresponding simple reflection. We set

$$s_\alpha(z) = (T, B^{s_\alpha}, v_1^{s_\alpha}, ..., v_{\operatorname{rank} G}^{s_\alpha}) \tag{2.19}$$

where

- $B^{s_\alpha}$ is the (unique) Borel subgroup of $G$ which contains $T$ and is in position $s_\alpha$ with $B$
- if $(\omega_i, \alpha) = 0$ then $v_i^{s_\alpha} = v_i$. If $(\omega_i, \alpha) = 1$ then $v_i^\alpha$ is the unique highest weight vector in $V\omega_i$ with respect to $B^{s_\alpha}$ which satisfies the following property. Let $\pi_{\mathcal{B},\alpha} :\ \mathcal{B} \to \mathcal{B}_\alpha$ denote the natural projection from $\mathcal{B}$ to the corresponding partial flag variety (thus the fibers of $\pi_{\mathcal{B},\alpha}$ are projective lines). Let $W_{B,\alpha}$ denote the space of global sections of the line bundle $\mathcal{O}(\omega_i)$ restricted to $\pi_{\mathcal{B},\alpha}^{-1}(B)$. Then $W_{B,\alpha}$ naturally identifies with the inverse image under $\overline{\pi}_\alpha$ of the image of $z$ in the space $X_\alpha$ (cf. 2.3.3 for all the notations). Therefore, $W_{B,\alpha}$ is a 2-dimensional vector space with has a natural symplectic form $\omega$ (recall, that we have fixed a $G$-invariant symplectic form on the bundle $\overline{\pi}_\alpha$. Let $w, w^{s_\alpha}$ be the images of $v_i$ and $v_i^{s_\alpha}$ in $W_{B,\alpha}$ respectively. Then $V_i^{s_\alpha}$ is uniquely determined by the requirement

$$\omega(w, w^{s_\alpha}) = 1 \tag{2.20}$$

It is easy to see that $Z$ satisfies the three conditions, listed above. In particular, the $G$-action on $Z$ is simply transitive (this follows immediately from the fact that

$Z_{\mathcal{B}}$ is a homogeneous space over $G$, where stabilizer of a point is a Cartan subgroup, and, on the other hand, $X$ is a $T$-torsor over $\mathcal{B}$).

2.4.2. *The spaces $Z_w$.* For any $w \in W$ we let $\Gamma_w$ denote the graph $w$ on $Z$. We denote now by $Z_w$ the image of $\Gamma_w$ in $X \times X$ (under the map $p \times p$).

**Proposition 2.** *1) $Z_w$ is a smooth closed subvariety of $X \times X$ of dimension equal to* $\dim X + l(w)$.

*2) The restrictions of the projections $\mathrm{pr}_1, \mathrm{pr}_2$ (going from $X \times X$ to $X$) to $Z_w$ are locally trivial (in étale topology) fibrations with fiber isomorphic to $\mathbb{A}^{l(w)}$.*

*3) $Z_e$ is equal to the diagonal in $X \times X$ and $Z_{w_0}$ is isomorphic to $Z$, where the two projections from $Z$ to $X$ are given by $p$ and $p \circ w_0$ (here $w_0$ denotes the longest element in $W$). Also, for any simple reflection $s_\alpha \in W$ we have $Z_{s_\alpha} = Z_{\overline{X}^\alpha}$ (recall (cf.* 2.3.3*) that $\overline{\pi}_\alpha : \overline{X}^\alpha \to X_\alpha$ is a 2-dimensional symplectic vector bundle, whose complement to the zero section is naturally identified with $X$. Recall also that in* 2.1.2 *we have defined certain variety $Z_E$ (the kernel of the Radon transformation) for any vector bundle $E$).*

*4) Suppose that $G$ is defined over $\mathbb{F}_q$ and let $\mathrm{Fr} : X \to X$ denote the corresponding Frobenius morphism (any $\mathbb{F}_q$-rational structure on $G$ gives rise to a canonical $\mathbb{F}_q$-rational structure on $X$). Let $\Gamma_{\mathrm{Fr}}$ denote the graph of the Frobenius morphism on $X$. Then the intersection of $Z_w$ with $\Gamma_{\mathrm{Fr}}$ is transverse.*

*5) Let $w_1, w_2 \in W$ be such that $l(w_1 w_2) = l(w_1) + l(w_2)$. Then $Z_{w_1} \circ Z_{w_2} = Z_{w_1 w_2}$, where "$\circ$" denotes composition of correspondences.*

The proof of the Proposition is straightforward and it is left to the reader.

2.4.3. *Radon transforms.* Let $\mathrm{pr}_{1,2} : X \times X \to X$ denote again the two natural projections and let $K_w = \overline{\mathbb{Q}}_{\ell, Z_w}[l(w)] \in \mathcal{D}(X \times X)$. Then we define the functor $\mathcal{R}_w : \mathcal{D}(X) \to \mathcal{D}(X)$ by

$$\mathcal{R}_w(A) = \mathrm{pr}_{2!}(\mathrm{pr}_1^* A \otimes K_w) \tag{2.21}$$

**Proposition 3.** *1) Let $w_1, w_2 \in W$ be such that $l(w_1 w_2) = l(w_1) + l(w_2)$. Then there is a canonical isomorphism of functors*

$$\mathcal{R}_{w_1} \circ \mathcal{R}_{w_2} \to \mathcal{R}_{w_1 w_2} \tag{2.22}$$

*These isomorphisms extend the functors $\mathcal{R}_w$ to an action of the braid monoid corresponding to $W$ on the category $\mathcal{D}(X)$ in the sense of Deligne ([5]).*

*2) The functors $\mathcal{R}_w$ map the category $\mathcal{D}^{mon}(X)$ to itself. Moreover, for any simple reflection $s_\alpha \in W$ the restrictions of the functors $\mathcal{R}_{s_\alpha}$ and $\mathcal{F}_{s_\alpha}$ to the category $\mathcal{D}^{mon}(X)$ are canonically isomorphic.*

*Proof.* The proof of the first statement follows merely from point 5 in Proposition 2.4.2. The proof of the second statement follows from Theorem 4 and from the fact that $Z_{s_\alpha}$ is equal to $Z_{\overline{X}^\alpha}$ (cf. Proposition 2.4.2(3)). $\square$

**Corollary 3.** *Theorem* 5 *holds.*

For any $w \in W$ we will denote by $\Phi_w : \mathcal{D}^0(X) \to \mathcal{D}^0(X)$ the corresponding functor (here we use the fact that $W$ has a natural set-theoretical embedding into $\mathrm{Br}_W$). One can show (cf. [19]) that the functors $\Phi_w$ extend canonically to an action of the group $W$ on the category $\mathcal{D}^0(X)$. However, we will not use this in the sequel.

**2.5. Remarks on general reductive groups.** So far we were speaking only about connected simply connected semisimple groups $G$. Let us explain how to extend the above construction to the case of arbitrary connected reductive group $G$. First of all, suppose that $G$ is semisimple. Let $G^{sc} \to G$ denote the universal covering of $G$ and let $Z(G^{sc})$ denote the center of $G^{sc}$, which is the Galois group of the covering $G^{sc} \to G$. Then we can repeat all the constructions and statements discussed above, replacing everywhere the group $G$ by $G^{sc}$ and the words sheaf on $G/U$" by the words "$Z(G^{sc})$ -sheaf on $G^{sc}/U$". The case of general reductive group can be treated similarly.

## 3. Character sheaves

This section is devoted to some preliminary material about character sheaves, that we are going to need in section 5. Below $G$ is a reductive algebraic group defined over $\mathbb{F}_q$.

**3.1. Maximal tori and their characters.**

3.1.1. *Maximal tori and conjugacy classes in the Weyl group.* In this section we let $T$ denote the abstract Cartan group of $G$ with the corresponding $\mathbb{F}_q$-rational structure (cf. [6], Section 1.1). Let $W$ denote the corresponding Weyl group, which is endowed with the action of the corresponding Frobenius automorphism $\mathrm{Fr}:\ W \to W$. Let $W_F^{\#}$ denote the set of Fr-conjugacy classes of elements in $W$ (by definition, two elements $w_1$ and $w_2$ are Fr-conjugate if there exists $y \in W$ such that $w_1 = yw_2\mathrm{Fr}(y)^{-1}$).

It is shown in [6] that there is a natural bijection between $W_F^{\#}$ and the set of conjugacy classes of maximal tori in $G$ which are defined over $\mathbb{F}_q$. For any $w \in W$ let $T_w$ denote the corresponding torus (defined uniquely up to $G(\mathbb{F}_q)$-conjugacy). As an algebraic group over $\overline{\mathbb{F}}_q$ the torus $T_w$ is identified (canonically) with $T$. Let $\mathrm{Fr}_w :$ $T \to T$ denote the image of the Frobenius morphism of $T_w$ under this identification. Thus it follows from the definition of $T_w$ (cf. [6], Section 1.8) that

$$\mathrm{Fr}_w = w \circ \mathrm{Fr} \tag{3.1}$$

3.1.2. *Rigidified local systems.* Here we follow the exposition of G. Laumon (cf. [12]). Recall that $T$ denotes the abstract Cartan group of $G$ with the split $\mathbb{F}_q$-structure. Let $\mathcal{S}(T)$ denote the set of isomorphisms of couples $(\mathcal{L}, \iota)$, where $\mathcal{L}$ is an $\ell$-adic one-dimensional local system on $T$ and $\iota:\ \overline{\mathbb{Q}}_\ell \widetilde{\to} \mathcal{L}_e$ is a rigidification of $\mathcal{L}$ (here $\mathcal{L}_e$ denotes the fiber of $\mathcal{L}$ at $e \in T$), such that the following property holds.

**Property.** There exists a natural number $n$ such that the pair $(\mu_n^*\mathcal{L}, \mu_n^*\iota)$ is isomorphic to the pair $(\overline{\mathbb{Q}}_{l,T}, 1)$, where $\mu_n:\ T \to T$ is given by $\mu_n(t) = t^n$, $\overline{\mathbb{Q}}_{l,T}$ is the constant sheaf on $T$ and $1:\ \overline{\mathbb{Q}}_\ell \to (\overline{\mathbb{Q}}_{l,T})_e$ is the obvious map.

The set $\mathcal{S}(T)$ has canonical structure of an abelian group, given by tensor product of $\overline{\mathbb{Q}}_\ell$-sheaves. One has a non-canonical isomorphism

$$\mathcal{S}(T) \simeq X^*(T) \otimes (\mathbb{Q}'/\mathbb{Z}) \tag{3.2}$$

where $X^*(T)$ is the group of algebraic characters of $T$ and

$$\mathbb{Q}' = \{\frac{m}{n} \in \mathbb{Q}|\ m, n \in \mathbb{Z} \text{ and } n \text{ is invertible in } \mathbb{F}_q\} \tag{3.3}$$

In the sequel we will write simply $\mathcal{L} \in \mathcal{S}(T)$, omitting $\iota$ (if it does not lead to a confusion).

The group $W$ acts naturally on the group $\mathcal{S}(T)$ (an element $w \in W$ sends $\mathcal{L}$ to $(w^{-1})^*\mathcal{L}$). For any $\mathcal{L} \in \mathcal{S}(T)$ we let $W_{\mathcal{L}}$ denote the stabilizer of $\mathcal{L}$ in $W$ (it is easy to see that $W_{\mathcal{L}}$ is independent of the rigidification).

**Definition 5.** *(cf.* [12]*)*

*1) An element $\mathcal{L} \in \mathcal{S}(T)$ is called regular if $W_{\mathcal{L}} = 1$.*

*2) An element $\mathcal{L} \in \mathcal{S}(T)$ is called quasi-regular if for any coroot $\alpha : \mathbb{G}_m \to T$ of $G$ the $\overline{\mathbb{Q}}_\ell$-sheaf $\alpha^*\mathcal{L}$ on $\mathbb{G}_m$ is not constant.*

**Example.** Let us give an example of a quasi-regular element of $\mathcal{S}(T)$, which is not regular. Let $G = SL(2)$. In this case $T$ is one-dimensional. Let $\tau : T \to T$ be given by $\tau(t) = t^2$. Consider $\tau_*(\overline{\mathbb{Q}}_{\ell,T})$. This sheaf admits a natural action of $\mathbb{Z}_2$ (which is the Galois group of the covering $\tau$). Let $\mathcal{L}$ denote the skew-invariants of $\mathbb{Z}_2$ on $\tau_*(\overline{\mathbb{Q}}_{\ell,T})$. Then $\mathcal{L}$ is a non-trivial one-dimensional local system on $T$ which admits an obvious rigidification. It is easy to see that $\mathcal{L}^{\otimes 2} \simeq \overline{\mathbb{Q}}_{\ell,T}$. Therefore, $\mathcal{L}$ defines an element in $\mathcal{S}(T)$. Since $\mathcal{L}$ is clearly non-constant, it follows that $\mathcal{L}$ is quasi-regular. However, since $\tau$ is a $W$-equivariant map, it follows that $w^*\mathcal{L} \simeq \mathcal{L}$ for any $w \in W$. Hence $\mathcal{L}$ is not regular.

However it follows essentially from [6], Theorem 5.13 that if the center of $G$ is connected then the notions of regularity and quasi-regularity coincide.

3.1.3. *Local systems and characters.* Suppose now that our torus $T$ is endowed with some $\mathbb{F}_q$-rational structure. Then the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts naturally on $\mathcal{S}(T)$. Let $\mathrm{Fr} : \mathcal{S}(T) \to \mathcal{S}(T)$ denote the corresponding Frobenius morphism. Suppose now that we are given some $(\mathcal{L}, \iota) \in \mathcal{S}(T)^{\mathrm{Fr}}$. Then $\mathcal{L}$ acquires a canonical structure of a Weil sheaf on $T$. Indeed, the fact that $(\mathcal{L}, \iota) \in \mathcal{S}(T)^{\mathrm{Fr}}$ means that there exists an isomorphism $\mathcal{L} \simeq \mathrm{Fr}^*\mathcal{L}$ which commutes with $\iota$ and it is easy to see that such automorphism is automatically unique. Thus for any $(\mathcal{L}, \iota) \in \mathcal{S}(T)^{\mathrm{Fr}}$ we may consider the corresponding trace function $\mathrm{tr}(\mathcal{L})$ on $T(\mathbb{F}_q)$.

**Lemma 2.** $\mathrm{tr}(\mathcal{L})$ *is a character of the finite group $T(\mathbb{F}_q)$.*

**3.2. Matrix coefficients sheaves.** Let now $G$ be an arbitrary algebraic group over $\overline{\mathbb{F}}_q$, $X$ – a $G$-space. Take any $A \in \mathcal{D}(X \times X)$. Then we can define the matrix

coefficient sheaf $m(A)$ by

$$m(A) = \pi_! i^* A \tag{3.4}$$

where $i : G \times X \to X \times X$ is defined by $i(g, x) = (gx, x)$ and $\pi : G \times X \to G$ is the projection on the first variable.

**3.3. Modified matrix coefficients.** Take now $G$ again to be a reductive group over $\overline{\mathbb{F}}_q$ and $X$ to be the basic affine space of $G$ and set $\mathcal{B} = X/T$ to be the flag variety of G. Let $\Delta : X \to X \times X$ be the diagonal embedding. In this section we are going to deal only with those sheaves on $X \times X$, which are $(T \times T, \mathcal{L} \boxtimes \mathcal{L}^{-1})$-equivariant for some one-dimensional local system $\mathcal{L}$ on $T$. In this case we are going to modify the definition of matrix coefficients. Namely, since the actions of $G$ and $T$ on $X$ commute, it follows that the sheaf $i^*A$ in formula (3.4) is $T$-equivariant with respect to the action of $T$ on the second variable in $G \times X$. Therefore, $i^*A$ is a pull-back of some sheaf $\widetilde{A}$ on $G \times \mathcal{B}$. Let $\widetilde{\pi} : G \times \mathcal{B} \to G$ be the projection on the second variable. We define the *modified matrix coefficient sheaf* by

$$\widetilde{m}(A) = \widetilde{\pi}_! \widetilde{A} \tag{3.5}$$

**3.4. Character sheaves.** Let us recall Lusztig's definition of (some of) the character sheaves. Let $\widetilde{G}$ denote the variety of all pairs $(B, g)$, where

- $B$ is a Borel subgroup of $G$
- $g \in B$

One has natural maps $\alpha : \widetilde{G} \to T$ and $\pi : \widetilde{G} \to G$ defined as follows. First of all, we set $\pi(B, g) = g$. Now, in order to define $\alpha$, let us remind that for any Borel subgroup $B$ of $G$ one has canonical identification $\mu_B : B/U_B \widetilde{\to} T$, where $U_B$ denotes the unipotent radical (in fact, this is how the abstract Cartan group $T$ is defined). Now we set $\alpha(B, g) = \mu_B(g)$.

Let $\mathcal{L} \in \mathcal{S}(T)$. We define $\mathcal{K}_{\mathcal{L}} = \pi_! \alpha^*(\mathcal{L})[\dim G]$. One knows (cf. [14], [12]) that the sheaf $\mathcal{K}_{\mathcal{L}}$ is perverse, and it is irreducible if $\mathcal{L}$ is regular. We want to rewrite this sheaf as certain (modified) matrix coefficient. Namely, let $X$ again be the basic affine space of $G$ and let $\widetilde{\Delta} \subset X \times X$ be the preimage in $X \times X$ of the diagonal in $\mathcal{B} \times \mathcal{B}$. Then one has canonical morphism $f : \widetilde{\Delta} \to T$ such that for any $(x, y) \in \widetilde{\Delta}$ one has $y = f((x, y))x$ (recall that $T$ acts on $X$.

Set now $\mathcal{A}_{\mathcal{L}} = f^*(\mathcal{L})$.

**Lemma 3.** *One has canonical isomorphism*

$$\widetilde{m}(\mathcal{A}_{\mathcal{L}})[\dim G] \simeq \mathcal{K}_{\mathcal{L}} \tag{3.6}$$

The proof is straightforward. However, we will see in the next section that this essentially trivial lemma helps us define a Weil structure on $\mathcal{K}_{\mathcal{L}}$ without appealing to the fact that it is the intermediate extension from the set of regular semisimple elements in $G$.

**3.5. The Weil structure.** Fix now an isomorphism $\mathcal{L} \simeq \mathrm{Fr}_w^*(\mathcal{L})$ (i.e. we suppose that $\mathcal{L} \in \mathcal{S}(T)^{\mathrm{Fr}_w}$). It was observed by G. Lusztig in [14] that fixing such an isomorphism endows $\mathcal{K}_{\mathcal{L}}$ canonically with a Weil structure. Lusztig's definition of this Weil structure was as follows.

Let $j : G_{rs} \to G$ denote the open embedding of the variety of regular semisimple elements in $G$ into $G$.

**Lemma 4.**

$$\mathcal{K}_{\mathcal{L}} = j_{!*}(\mathcal{K}_{\mathcal{L}}|_{G_{rs}}) \tag{3.7}$$

*Here $j_{!*}$ denotes the Goresky-McPherson (intermediate) extension (cf.* [1]*).*

(The lemma follows from the fact that the map $\pi$ is small in the sense of Goresky and McPherson).

The lemma shows that it is enough to construct the Weil structure only on the restriction of $\mathcal{K}_{\mathcal{L}}$ on $G_{rs}$. The latter now has a particularly simple form. Namely, let $\widetilde{G}_{rs}$ denote the preimage of $G_{rs}$ under $\pi$ and let $\pi_{rs}$ denote the restriction of $\pi$ to $\widetilde{G}_{rs}$. Then it is easy to see that $\pi_{rs} : \widetilde{G}_{rs} \to G_{rs}$ is an unramified Galois covering with Galois group $W$. In particular, $W$ acts on $\widetilde{G}_{rs}$ and this action is compatible with the action of $W$ on $T$ in the sense that the restriction of $\alpha$ on $\widetilde{G}_{rs}$ is $W$-equivariant.

Now, an isomorphism $\mathcal{L} \simeq \mathrm{Fr}_w^*(\mathcal{L})$ gives rise to an isomorphism

$$\alpha^*\mathcal{L} \simeq (w \circ \mathrm{Fr})^*(\alpha^*\mathcal{L}) \tag{3.8}$$

(here both $w$ and Fr are considered on the variety $\widetilde{G}$). Since $\pi_{rs}$ is a Galois covering with Galois group $W$, it follows that one has canonical identification

$$\pi_{rs!}(\mathrm{Fr}^*(\alpha^*\mathcal{L})) \simeq \pi_{rs!}((w \circ \mathrm{Fr})^*(\alpha^*\mathcal{L})) \tag{3.9}$$

Hence from (3.8) and (3.9) we get the identifications

$$\mathrm{Fr}^*\pi_{rs!}(\alpha^*\mathcal{L}) \simeq \pi_{rs!}(\mathrm{Fr}^*(\alpha^*\mathcal{L})) \simeq \pi_{rs!}((w \circ \mathrm{Fr})^*(\alpha^*\mathcal{L})) \simeq \pi_{rs!}(\alpha^*\mathcal{L}) \tag{3.10}$$

which gives us a Weil structure on $\pi_{rs!}(\alpha^*\mathcal{L}) \simeq \mathcal{K}_{\mathcal{L}}|_{G_{rs}}$. Hence we have defined a canonical Weil structure on $\mathcal{K}_{\mathcal{L}}$.

**3.6. The Weil structure revisited (the case of quasi-regular $\mathcal{L}$).** We now want to give a different construction of (the same) Weil structure on $\mathcal{K}_{\mathcal{L}}$ using lemma 3 and the functors $\Phi_w$. For simplicity we will assume here that $\mathcal{L}$ is quasi-regular, since this is the only case that we are going to use in the sequel (but, with slight modifications, the argument presented below works for any local system $\mathcal{L}$).

**Proposition 4.** *Let $\mathcal{L} \in \mathcal{S}(T)$ be quasi-regular. Then one has canonical isomorphism*

$$\Phi_{w,w} f^*(\mathcal{L}) \simeq f^*(w(\mathcal{L})) \tag{3.11}$$

*(recall that $w(\mathcal{L}) = (w^{-1})^*\mathcal{L}$).*

*Proof.* Since the notion of quasi-regularity is invariant under $W$, it is enough to show that (7.5) holds when $w$ is a simple reflection. In this case, arguing in a standard way we may assume that $G = SL(2)$. So, we must construct the isomorphism (7.5) when $G = SL(2)$, $w = s$ – the unique non-trivial element in the Weyl group of $SL(2)$ and $\mathcal{L}$ – any non-constant element from $\mathcal{S}(T) = \mathcal{S}(\mathbb{G}_m)$.

First of all, let us show that the restriction of $\Phi_{w,w} f^*(\mathcal{L})$ to the complement of $\widetilde{\Delta} \subset X \times X$ is equal to zero. Let $(x, y) \in (X \times X)$. Then the fiber of $\Phi_{w,w} f^*(\mathcal{L})$ at $(x, y)$ can be computed in the following way.

Recall that $X \simeq \mathbb{A}^2 \backslash \{0\}$ and let $Z$ denote the kernel of the Radon transform on $X$ (with respect to a fixed symplectic form $\omega$ – cf. Section 2). Let $Z_{x,y}$ denote the closed subvariety of $X \times X$, consisting of all pairs $(x_1, y_1) \in X \times X$ such that

- $(x_1, y_1) \in \widetilde{\Delta}$

- $(x_1, x) \in Z$
- $(y_1, y) \in Z$

Let $j: Z_{x,y} \to \widetilde{\Delta}$ be the embedding of $Z_{x,y}$ into $\widetilde{\Delta}$ (or $X \times X$). Then the fiber of $\Phi_{w,w} f^*(\mathcal{L})$ at $(x, y)$ is naturally isomorphic to $H_c^*(Z_{x,y}, j^*(f^*(\mathcal{L}))[1]$. Let $H_{x,y}$ denote the stabilizer of $(x, y)$ in $G \times T \times T$.

Suppose now that $(x, y) \notin \widetilde{\Delta}$. Then the restriction of $f$ to $Z_{x,y}$ is an isomorphism of the latter variety with $T = \mathbb{G}_m$. Indeed, without loss of generality we may suppose that $\omega(x, y) = 1$. Then if $f((x_1, y_1)) = \lambda$, i.e. $y_1 = \lambda x_1$, then the pair $(x_1, y_1)$ is given by the formulas

$$x_1 = \lambda^{-1} x + y, \quad y_1 = x + \lambda^{-1} y$$

Hence the pair $Z_{x,y}, j^*(f^*(\mathcal{L}))$ is isomorphic to the pair $\mathbb{G}_m, \mathcal{L}$ and therefore, since $\mathcal{L}$ is non-constant, we have

$$H_c^*(Z_{x,y}, j^*(f^*(\mathcal{L})) = H_c^*(\mathbb{G}_m, \mathcal{L}) = 0 \tag{3.12}$$

Hence the restriction of $\Phi_{w,w} f^*(\mathcal{L})$ to the complement of $\widetilde{\Delta}$ is equal to 0.

On the other hand, the restriction of $\Phi_{w,w} f^*(\mathcal{L})$ to $\widetilde{\Delta}$ is equal to $p_! p_1^*(f^*(\mathcal{L}))[1]$ where we have the diagram

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times X & & X \times X \end{tikzcd}$$

Here $W$ is the subvariety of $X^4$ consisting of all quadruples $(x_1, x_2, x, y)$, subject to the three conditions above and such that $(x, y) \in \widetilde{\Delta}$ and

$$p_1((x_1, y_1, x, y)) = (x_1, y_1), \quad p((x_, y_1, x, y)) = (x, y)$$

It is straightforward that $f \circ p_1 = s \circ f \circ p$ (recall that $s$ acts by $s(\lambda) = \lambda^{-1}$ and that $p$ is a smooth fibration with fiber $\mathbb{A}^1$. This implies immediately that

$$p_! p_1^*(f^*(\mathcal{L}))[1] = f^*(s^* \mathcal{L}) \tag{3.13}$$

which finishes the proof. $\square$

Let us now explain how the above proposition helps us define a Weil structure on $\mathcal{K}_{\mathcal{L}}$ (for quasi-regular $\mathcal{L}$). For this we need another auxiliary result.

**Proposition 5.** *Let $\mathcal{L}$ be as above and let $A \in \mathcal{D}^{\mathrm{reg}}(X \times X)$. Then for any $w \in W$ one has canonical isomorphisms*

$$m(A) \simeq m(\Phi_{w,w} A) \tag{3.14}$$

*If $A$ is $(T \times T, \mathcal{L} \boxtimes \mathcal{L}^{-1})$-equivariant for some quasi-regular $\mathcal{L} \in \mathcal{S}(T)$ then one also has canonical isomorphism*

$$\widetilde{m}(A) \simeq \widetilde{m}(\Phi_{w,w} A) \tag{3.15}$$

*Proof.* We will prove here that (3.14) holds. The proof of (3.15) is completely analogous.

For the sake of simplicity, let us construct a natural isomorphism of the fibers of $m(A)$ and $\Phi_{w,w}(A)$ at every point $g \in G$. Since the functor $\Phi_{w,w}$ "commutes" with the $G \times G$-action on $X \times X$, it is enough, in fact, to construct such an isomorphism for $g = e$ (the unit element), because the fiber of $m(A)$ at the point $g$ is canonically isomorphic to the fiber of $m((g \times e)^* A)$ at the point $e$. Let $\Delta : X \to X \times X$ denote the diagonal embedding of $X$ and let $\Delta X$ denote its image. Then we have to construct a canonical isomorphism between $H_c^*(\Delta^* A)$ and $H_c^*(\Delta^* \Phi_{w,w}(A))$.

Also, we may assume, without loss of generality that $w$ is a simple reflection $s_\alpha$ in $W$. It is easy to see that in this case our statement reduces immediately to the case $G = SL(2)$.

Let us now make an explicit calculation in this case. Recall that $X$ in this case is equal to $\mathbb{A}^2 \backslash \{0\}$, which is endowed with a symplectic form $\omega$, and we have also the variety $Z$, defined as

$$Z = \{(x, y) \in X \times X |\ \omega(x, y) = 1\}$$

together with two natural projections $p_1, p_2 :\ Z \to X$. Let now $A \in \mathcal{D}^{\mathrm{reg}}(X \times X)$. Then $\Phi_{w,w}(A) = (p_2 \times p_2)_!(p_1 \times p_1)^* A[2]$. Hence,

$$H_c^*(\Delta^* \Phi_{w,w}(A)) = H_c^*((p_2 \times p_2)^{-1}(\Delta X), (p_1 \times p_1)^* A)[2] \tag{3.16}$$

(here we denoted by the same symbol $p_1 \times p_1$ both the natural projection from $Z \times Z$ to $X \times X$ and its restriction to $(p_2 \times p_2)^{-1}(\Delta X)$).

Now, the variety $(p_2\times p_2)^{-1}(\Delta X)$ can be described as the set of all triples $(x,y,z)\in X^3$, such that $\omega(x,z)=1$ and $\omega(y,z)=1$ (the corresponding point in $Z\times Z$ is $((x,z),(y,z))$). Let now $q$ denote the restriction of $p_1\times p_1$ to $(p_2\times p_2)^{-1}(\Delta X)$. Then $q((x,y,z))=(x,y)$. Let us also denote $(p_2\times p_2)^{-1}(\Delta X)$ by $W$.

Let us compute $H^*_c(q^*A)$ by first applying the functor $q_!$ and then computing cohomology on $X\times X$. Namely, consider the sheaf $q_!A$. Denote by $j:\ Y\to X\times X$ the embedding of the complement to $\Delta X$ into $X$. Then we have an exact triangle

$$j_!j^*(q_!q^*A)\to q_!q^*A\to \Delta_!\Delta^*(q_!q^*A) \tag{3.17}$$

It is easy to see now that $\Delta^*(q_!q^*A)$ is naturally isomorphic to $\Delta^*A[-2]$, since the restriction of $q$ to $q^{-1}(\Delta X)$ is a fibration with fiber $\mathbb{A}^1$. Therefore, in order to construct an isomorphism between $H^*_c(\Delta^*A)$ and $H^*_c(\Delta^*\Phi_{w,w}(A))=H^*_c(q^*A)[2]$ it is enough to show that

$$H^*_c(j_!j^*(q_!q^*A))=0 \tag{3.18}$$

Recall now that $\widetilde{\Delta}$ denotes the variety of all pairs $(x,y)\in X\times X$, which lie on the same line. Let $\widetilde{Y}$ denote the complement of $\widetilde{\Delta}X$ in $X\times X$ and let $\widetilde{j}:\ \widetilde{Y}\to X\times X$ denote the corresponding embedding.

**Lemma 5.** *The canonical map* $\widetilde{j}_!\widetilde{j}^*(q_!q^*A)\to j_!j^*(q_!q^*A)$ *is an isomorphism.*

The lemma follows immediately from the fact that the fiber of $q$ over any point of $\widetilde{\Delta}X\backslash\Delta X$ is empty.

Now we can finish the proof. It follows from the lemma that it is enough for us to show that $H^*_c(\widetilde{j}^*q_!q^*A)=0$. It is easy to see that $\widetilde{Y}$ is invariant under the $\mathbb{G}_m\times\mathbb{G}_m$-action on $X\times X$. However, we have assumed that $A\in\mathcal{D}^{\mathrm{reg}}(X\times X)$. On the other hand, since over $\widetilde{Y}$ the map $q$ is an isomorphism (which can be easily checked), it follows that $\widetilde{j}^*q_!q^*A=\widetilde{j}^*A\in\mathcal{D}^{\mathrm{reg}}(\widetilde{Y})$, i.e. $\widetilde{j}^*q_!q^*A$ is glued from $(\mathbb{G}_m\times\mathbb{G}_m,\mathcal{L}_1\boxtimes\mathcal{L}_2)$- equivariant sheaves, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are nontrivial local systems on $\mathbb{G}_m$. But $H^*_c(\widetilde{Y},B)=0$ for any $B\in\mathcal{D}^{\mathrm{reg}}(\widetilde{Y})$, which finishes the proof. $\square$

We can now finish constructing the Weil structure on $\mathcal{K}_{\mathcal{L}}$. Indeed, we have

$$\begin{gathered}\mathrm{Fr}^*\mathcal{K}_{\mathcal{L}}\simeq\mathrm{Fr}^*\widetilde{m}(\mathcal{A}_{\mathcal{L}})\simeq\widetilde{m}(A(\mathrm{Fr}^*\mathcal{L}))\simeq\\ \widetilde{m}(A(w(\mathcal{L})))\simeq\widetilde{m}\Phi_{w,w}(\mathcal{A}_{\mathcal{L}})\simeq\widetilde{m}(\mathcal{L})\simeq\mathcal{K}_{\mathcal{L}}\end{gathered} \tag{3.19}$$

Hence we get a Weil structure on $\mathcal{K}_{\mathcal{L}}$.

**3.7. Some other character sheaves.** It is easy to see that the $G\times T\times T$-orbits on $X\times X$ are naturally parametrized by $W$. For $w\in W$ let $\mathcal{O}_w$ denote the corresponding orbit.

For any $\mathcal{L}_1,\mathcal{L}_2\in\mathcal{S}(T)$ define

$$W_{\mathcal{L}_1,\mathcal{L}_2}=\{w\in W|\ w(\mathcal{L}_1)=\mathcal{L}_2\} \tag{3.20}$$

Thus $W_{\mathcal{L}}=W_{\mathcal{L},\mathcal{L}}$.

In the sequel we will need the following result.

**Proposition 6.** *1) Let $\mathcal{L}_1,\mathcal{L}_2\in\mathcal{S}(T)$. Then for every $y\in W_{\mathcal{L}_1,\mathcal{L}_2}$ there is unique (up to isomorphism) simple $(G\times T\times T,\overline{\mathbb{Q}}_{\ell,G}\boxtimes\mathcal{L}_1\boxtimes\mathcal{L}_2^{-1})$-equivariant perverse sheaf $\mathcal{A}^y_{\mathcal{L}_1,\mathcal{L}_2}$ supported on the closure of $\mathcal{O}_w$. The sheaves $\mathcal{A}^y_{\mathcal{L}_1,\mathcal{L}_2}$ form a basis of the Grothendieck group (tensored with $\overline{\mathbb{Q}}_\ell$) of the category of $(G\times T\times T,\overline{\mathbb{Q}}_{\ell,G}\boxtimes\mathcal{L}_1\boxtimes\mathcal{L}_2^{-1})$- equivariant perverse sheaves on $X\times X$ (or the Grothendieck group of the corresponding derived category). Therefore, the dimension of this group is equal to $\#W_{\mathcal{L}_1,\mathcal{L}_2}$.*

*2) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are quasi-regular, then for every $w_1,w_2\in W$ the sheaf $\Phi_{w_1,w_2}(\mathcal{A}^y_{\mathcal{L}_1,\mathcal{L}_2})$ is isomorphic to $\mathcal{A}^{w_1^{-1}yw_2}_{w_1(\mathcal{L}_1),w_2(\mathcal{L}_2)}$.*

*Proof.* For the proof of 1) it is enough to note that $\mathcal{O}_y$ carries a non-zero $(G\times T\times T,\overline{\mathbb{Q}}_{\ell,G}\boxtimes\mathcal{L}_1\boxtimes\mathcal{L}_2^{-1})$ equivariant local system if and only if $y\in W_{\mathcal{L}_1,\mathcal{L}_2}$.

Let us prove 2). Clearly, it is enough to prove 2) when is the pair $(w_1,w_2)$ is of the form $(s,e)$ or $(e,s)$ , where $e\in W$ is the unit element and $s=s_\alpha$ is a simple reflection. In this case the statement of 2) easily reduces to the case $G=SL(2)$ and thus can be checked by an explicit calculation (as in the proof of Proposition 4). $\square$

We will denote the sheaf $\mathcal{A}^y_{\mathcal{L},\mathcal{L}}$ just by $\mathcal{A}^y_{\mathcal{L}}$. Thus $\mathcal{A}_{\mathcal{L}}=\mathcal{A}^e_{\mathcal{L}}$, where $e\in W$ is the unit element.

**Corollary 4.** *Suppose that $\mathcal{L}\in\mathcal{S}(T)^{Fr_w}$ and that $\mathcal{L}$ is quasi-regular. Then the perverse sheaf $\Phi_{w,w}(Fr^*\mathcal{A}^y_{\mathcal{L}})$ is naturally isomorphic to $\mathcal{A}^{Fr_w y}_{\mathcal{L}}$.*

The proof is straightforward (one should apply Proposition 6 for $\mathcal{L}_1=\mathcal{L}_2=\mathcal{L}$ and $w_1=w_2=w$).

In the sequel we will need also the following notation. Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}(T)$ and let $y \in W_{\mathcal{L}_1,\mathcal{L}_2}$. Then together with the sheaf $\mathcal{A}^y_{\mathcal{L}_1,\mathcal{L}_2}$ we will consider also the sheaves $\mathcal{A}^{y,!}_{\mathcal{L}_1,\mathcal{L}_2}$ and $\mathcal{A}^{y,*}_{\mathcal{L}_1,\mathcal{L}_2}$, which are respectively "!" and "$*$" extensions of the corresponding $(G \times T \times T, \overline{\mathbb{Q}}_{\ell,G} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{L}_2^{-1}$-equivariant local system on $\mathcal{O}_y$ to the whole of $X \times X$. Thus we have natural morphisms

$$\mathcal{A}^{y,!}_{\mathcal{L}_1,\mathcal{L}_2} \to \mathcal{A}^{y}_{\mathcal{L}_1,\mathcal{L}_2} \to \mathcal{A}^{y,*}_{\mathcal{L}_1,\mathcal{L}_2} \tag{3.21}$$

It is easy to deduce from [14] that all character sheaves are by definition, those perverse sheaves which occur as constituents of sheaves of the form $\widetilde{m}(A)$, where $A$ is $(G \times T \times T, \overline{\mathbb{Q}}_{\ell,G} \boxtimes \mathcal{L} \boxtimes \mathcal{L}^{-1})$- equivariant. However, we will not need this in the sequel.

## 4. Kazhdan-Laumon representations: statement of the results

In this section we are going to define our version of Kazhdan-Laumon representations and state our main results about them. However, first we want to explain certain general ideas (which are due to D. Kazhdan – cf. [10]) which hide behind this definition. We will explain these general ideas in the case when $G$ is a group over a local field, but then come back to finite fields again.

**4.1. Forms of principal series: general ideas.** In this section $k$ will be a non-archimedian local field. For any algebraic variety $Y$ over $k$ we will denote by $Y(k)$ its set of $k$-rational points. We will assume that our $G$ is a semisimple connected, simply connected split group over $k$. It is an old ideology (going back to I. M. Gelfand) that to every maximal torus $T$ in $G$, defined over $k$, there should correspond certain "series" of representations of $G(k)$, parameterized by the characters of $T(k)$. Existence of such series is also predicted by the Langlands local reciprocity law. What do we mean by series? In fact, what one would like to construct is some canonical smooth representation of the group $G(k) \times T(k)$ (and then, given a character of $T(k)$, we can construct a representation of $G(k)$ taking the coinvariants of $T(k)$ with respect to this character). We will denote this (desired) representation by $V_{T,k}$.

There is now one case when one (almost) knows the answer for $V_{T,k}$. This is the case when $T$ is split over $k$. In this case we can take as $V_{T,k}$ just the corresponding space of principal series representations. I.e. let $X = G/U$ be the basic affine space of

$G$, where $U$ is a maximal unipotent subgroup of $G$ defined over $k$. Then $X$ is a quasi-affine algebraic variety, defined over $k$, and we take $V_{T,k} = \mathcal{S}(X(k))$ – the space of locally constant compactly supported functions on the set $X(k)$ of $k$–rational points of $X$ (note that $X(k) = G(k)/U(k)$). Since $G(k)$ acts naturally (on the left) on $X(k)$ and $T(k)$ acts there naturally on the right ($T$ is a torus in $G$ which normalizes $U$), our space is a representation of $G(k) \times T(k)$.

What can we do for other tori? Let us describe one idea in this direction (cf. [10]). Galois theory tells us that different conjugacy classes of maximal tori in $G(k)$ are classified by the homomorphisms $\Gamma \to W$ (up to $W$-conjugacy) where $\Gamma$ is the absolute Galois group of $k$ and $W$ is the Weyl group of $G$. Now one would like to think about different series of representations of $G(k)$ as "forms" of the principal series. Let us see how we can do it. Let $T$ be a torus in $G$ defined over $k$. We will assume the existence of $V_{T,k}$ and try to see what it implies. The Langlands reciprocity law tells us that for any Galois extension $k'/k$ there should exist a "lifting" of $V_{T,k}$ to a representation of $G(k') \times T(k')$ on which the group $\mathrm{Gal}(k'/k)$ acts, and this representation should be isomorphic to $V_{T,k'}$. Suppose now that $T$ splits over $k'$. Then we must get some non-trivial action of $\mathrm{Gal}(k'/k)$ on $\mathcal{S}(X(k')) = V_{T,k'}$. How to construct it? Since $T$ splits over $k'$, it corresponds to a homomorphism $\pi : \mathrm{Gal}(k'/k) \to W$. Therefore, it is enough for us to describe an action of the group $W$ on $\mathcal{S}(X(k'))$ (and then twist the obvious action of $\mathrm{Gal}(k'/k)$ on this space (coming from its action on $X(k')$) by means of $\pi$). This is already an "algebraic" problem (i.e. it has nothing to do with field extensions), and so, it is enough to describe the $W$-action on the space $\mathcal{S}(X(k))$. Unfortunately, the space $\mathcal{S}(X(k))$ does not admit any natural action of $W$. However, it is well-known that the space $L^2(X(k), \mu)$ does, where $\mu$ is a $G(k)$-invariant measure on $X(k)$. Consider, for simplicity, the example $G = SL(2)$. Then $X = \mathbb{A}^2\backslash\{0\}$ and $L^2(X(k)) = L^2(k^2)$. Also $W = \pm 1$. The space $k^2$ admits unique up to a scalar $SL(2,k)$-invariant symplectic form $\omega$. Let $-1 \in W$ act by the Fourier transform $F$ on $L^2(k^2)$ where we identify $k^2$ with the dual vector space by means of $\omega$. Then $F^2 = id$ (because $\omega$ is symplectic) and this is the desired action. In the general case the corresponding action can be described just repeating the construction described in 2.3 and replacing everywhere sheaves by functions (cf. [10]).

The first question which immediately appears here is the following

**Question 1.** Find a "nice" $W$-invariant subspace of $L^2(X(k))$ which contains $\mathcal{S}(X(k))$. This is an important question in itself. In particular, the definition of the above space should make sense in the adelic situation as well, where it might be used to study analytic properties of Eisenstein series. Also this space should give the "correct" $W$-equivariant version of principal series (this is why I wrote that one "almost" knows the answer for $V_{T,k}$ in the case of a split torus). Of course, one could take just the minimal subspace with the above properties. But such a definition is very difficult to work with (in particular, it does not make sense in the global situation). On the other hand for $G = SL(2)$ this space is just the space $\mathcal{S}(k^2)$ of smooth compactly supported functions on $k^2$. One would like to have a similar description of this space for any $G$. We will not discuss a solution of this question in this paper. Instead, (in the case of finite fields) we will restrict ourselves to some smaller space, on which $W$ will act. This space will be roughly "the space of all trace functions of Weil sheaves lying in $\mathcal{D}^0(X)$").

But even if we can answer Question 1, then we arrive to a much more intriguing

**Question 2.** How to construct the representations $V_{T,k}$ which would satisfy all the above properties? At the moment an answer is given only in the case when we replace our field $k$ by a finite field (in that case, of course, one has the Deligne-Lusztig construction of representations, but in view of what was said above one would like to have a different construction, which is more "compatible" with the theory of forms of algebraic objects. A discussion of such a construction is presented in 2.2). In the $p$-adic case D. Kazhdan has given some conjectural way to construct forms of the principal series for $G = GL(n)$ (however, he did not prove that these forms are really well-defined – cf. [10].

**4.2. Notations.** Now we come back to the case of finite field. ¿From now on in this section we will suppose that the group $G$ is defined over $k = \mathbb{F}_q$. Our purpose here is to define certain representations $V_{\mathcal{L},w}$ of the finite group $G(\mathbb{F}_q)$, where $\mathcal{L} \in \mathcal{S}(T)^{\mathrm{Fr}_w}$ for some $w \in W$ (moreover we will see that, in fact, $V_{\mathcal{L},w}$ depends only on $\mathcal{L}$ and not on $w$). The definition will be a slight modification of that from [11]. We hope that the connection of the definition with the general discussion of the previous section will be clear.

In the definitions below will work with Grothendieck groups of various abelian

categories of perverse sheaves on $X$, endowed with additional structures. However, one can also work with the Grothendieck group of the corresponding derived category, which would (trivially) lead to the same answer.

### 4.3. Kazhdan-Laumon representations.

4.3.1. *Some motivation.* Before giving the definition, let us explain some general fact (which is quite well-known – cf. [11]). This fact will serve as a motivation for the definition. Namely, following the ideas, described in 4.1 we need to define "forms" of the space of functions on $X(\mathbb{F}_q)$. For this we need to describe a more algebraic definition of this space.

Let $X$ be an arbitrary scheme of finite type over $\mathbb{F}_q$ and let $L(X(\mathbb{F}_q))$ denote the space of functions on the finite set $X(\mathbb{F}_q)$. We want to give some purely algebro-geometric construction of this finite-dimensional space, which will not appeal also to the notion of $\mathbb{F}_q$-rational point of $X$.

Let $K(X)$ denote the Grothendieck group of perverse sheaves on $X$, tensored with $\overline{\mathbb{Q}}_\ell$ (which is the same as the Grothendieck group of the corresponding derived category, tensored with $\overline{\mathbb{Q}}_\ell$. However, for simplicity, we prefer to work with abelian categories, and not with triangulated ones). We have a natural surjective map $\mathrm{tr}: K(X) \to L(X(\mathbb{F}_q))$. We would like to identify the kernel of this map. This is done as follows. One can define a canonical $\overline{\mathbb{Q}}_\ell$-valued symmetric pairing $\langle\cdot,\cdot\rangle : K(X)\otimes K(X) \to \overline{\mathbb{Q}}_\ell$ in the following way. Let $(A, \alpha : A \to \mathrm{Fr}^*A)$, $(B, \beta : B \to \mathrm{Fr}^*B)$ be two Weil (perverse) sheaves on $X$. Then we have an induced endomorphism $\phi(\alpha,\beta) : \mathrm{RHom}(A, DB) \to \mathrm{RHom}(A, DB)$ defined as the composition

$$\mathrm{RHom}(A, DB) \to \mathrm{RHom}(\mathrm{Fr}^*A, \mathrm{Fr}^*DB) \to \mathrm{RHom}(A, DB) \tag{4.1}$$

and we define

$$\langle (A,\alpha), (B,\beta)\rangle = \sum_i (-1)^i \,\mathrm{Tr}(\phi(\alpha,\beta), \mathrm{Ext}^i(A, DB)) \tag{4.2}$$

It is clear, that$\langle\cdot,\cdot\rangle$ descends to a well-defined pairing on $K(X)$. The following result can be easily deduced from the Grothendieck-Lefschetz trace formula for the Frobenius correspondence.

**Proposition 7.** *The kernel of* $\mathrm{tr} : K(X) \to L(X(\mathbb{F}_q))$ *is equal to the kernel of the pairing* $\langle \cdot, \cdot \rangle$.

Hence we obtain a canonical isomorphism between $L(X(\mathbb{F}_q))$ and $K(X)/\operatorname{Ker}\langle \cdot, \cdot \rangle$.

4.3.2. *Definition of Kazhdan-Laumon representations.* Let now $\mathcal{L} \in \mathcal{S}(T)^{\mathrm{Fr}_w}$, i.e. $\mathcal{L}$ is a local system on $T$ such that $\mathrm{Fr}_w^* \mathcal{L}$ is isomorphic to $\mathcal{L}$. Our goal is to define certain $\ell$-adic representation $V_{\mathcal{L},w}$ of the group $G(\mathbb{F}_q)$ (we will see afterwards that $V_{\mathcal{L},w}$ will actually depend only on $\mathcal{L}$ and not on $w$). We will use the notations of 2.3.

Let $\mathrm{Perv}^0_{\mathcal{L},w}(X)$ denote the category, whose objects are pairs $(A, \alpha)$, where

- $A$ is a $(T, \mathcal{L})$-equivariant perverse sheaf on $X$, which lies in $\mathrm{Perv}^0(X)$ (as an abstract perverse sheaf)
- $\alpha : A \simeq \Phi_w(\mathrm{Fr}^* A)$ is an isomorphism

(note that an isomorphism $\mathcal{L} \simeq \mathrm{Fr}^* \mathcal{L}$ endows $\Phi_w(\mathrm{Fr}^* A)$ with the structure of $(T, \mathcal{L})$-equivariant sheaf).

Morphisms in the category $\mathrm{Perv}^0_{\mathcal{L},w}(X)$ are morphisms between sheaves, which commute with $w$. Note that if $(A, \alpha) \in \mathrm{Perv}^0_{\mathcal{L},w}(X)$ then $(DA, D(\alpha)^{-1}) \in \mathrm{Perv}^0_{\mathcal{L}^{-1},w}(X)$ (here we use the fact that $\Phi_w$ commutes with Verdier duality). The following lemma is easy (it follows from the exactness of the functor $\Phi_w$ on the category $\mathrm{Perv}^0(X)$).

**Lemma 6.** *The category* $\mathrm{Perv}^0_{\mathcal{L},w}(X)$ *is abelian.*

It is easy to see that the category $\mathrm{Perv}^0_{\mathcal{L},w}(X)$ admits a natural action of the group $G(\mathbb{F}_q)$ (coming from the geometric action of this group on $X$).

Set now $K_{\mathcal{L},w} = K(\mathrm{Perv}^0_{\mathcal{L},w}(X)) \otimes \overline{\mathbb{Q}}_\ell$ (here $K(\mathrm{Perv}^0_{\mathcal{L},w}(X))$ is the Grothendieck group of the category $\mathrm{Perv}^0_{\mathcal{L},w}(X)$. This is an infinite-dimensional vector space over $\overline{\mathbb{Q}}_\ell$, on which the finite group $G(\mathbb{F}_q)$ acts. We now want to define certain quotient of this $G(\mathbb{F}_q)$-representation, which will already be finite-dimensional.

First of all, we claim that there is a natural $G(\mathbb{F}_q)$-invariant pairing $\langle \cdot, \cdot \rangle$ between $K_{\mathcal{L},w}$ and $K_{\mathcal{L}^{-1},w}$. It is constructed in the following way. Let $(A, \alpha) \in \mathrm{Perv}^0_{\mathcal{L},w}(X)$ and $(B, \beta) \in \mathrm{Perv}^0_{\mathcal{L}^{-1},w}(X)$. Consider $\mathrm{RHom}(A, DB)$ (the RHom is computed just in the category $\mathcal{D}(X)$). Then we have an induced endomorphism $\phi(\alpha, \beta) : \mathrm{RHom}(A, DB) \to \mathrm{RHom}(A, DB)$ defined as the composition

$$\mathrm{RHom}(A, DB) \to \mathrm{RHom}(\Phi_w(A), \Phi_w(DB)) \to \mathrm{RHom}(A, DB) \tag{4.3}$$

*Remark* 4. Note that in section 6 we will use $\phi(\alpha,\beta)$ to denote a slightly different endomorphism.

Set now

$$\mathrm{Tr}(\phi(\alpha,\beta)) = \sum_i (-1)^i \,\mathrm{Tr}(\phi(\alpha,\beta), \mathrm{Ext}^i(A, DB)) \tag{4.4}$$

We now define $\langle (A,\alpha),(B,\beta)\rangle = \frac{\mathrm{Tr}(\phi(\alpha,\beta))}{\# T_w(\mathbb{F}_q)}$. It is clear that $\langle\cdot,\cdot\rangle$ descends to a well-defined pairing between $K_{\mathcal{L},w}$ and $K_{\mathcal{L}^{-1},w}$, which is $G(\mathbb{F}_q)$-equivariant. Denote now by $K^{\mathrm{null}}_{\mathcal{L},w}$ the left kernel of the pairing $\langle\cdot,\cdot\rangle$, i.e.

$$K^{\mathrm{null}}_{\mathcal{L},w} = \{a \in K_{\mathcal{L},w} |\ \langle a,b\rangle = 0 \text{ for any } b \in K_{\mathcal{L}^{-1},w}\} \tag{4.5}$$

Set now $V_{\mathcal{L},w} = K_{\mathcal{L},w}/K^{\mathrm{null}}_{\mathcal{L},w}$. It is clear that the group $G(\mathbb{F}_q)$ acts on $V_{\mathcal{L},w}$. However, it is not clear *a priori* whether this representation of $G(\mathbb{F}_q)$ possesses any good properties (for example, it is not obvious that $V_{\mathcal{L},w}$ is finite-dimensional). The following theorem says, in a sense, that it is really the case. It is one of the main results of this paper.

**Theorem 6.** *1) $V_{\mathcal{L}}$ is finite-dimensional. Moreover, if $\mathcal{L}$ is quasi-regular, then*

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(V_{\mathcal{L},w}, V_{\mathcal{L},w}) = \# W_{\mathcal{L}}^{\mathrm{Fr}_w} \tag{4.6}$$

*Hence, if $\mathcal{L}$ is regular, then $V_{\mathcal{L}}$ is irreducible.*

*2) Suppose that $\mathcal{L}$ is quasi-regular. Then the character of $V_{\mathcal{L}}$ is equal to* $\mathrm{tr}(\mathcal{K}_{\mathcal{L}})$ *(cf. subsection* 1.3*)*

*Remark* 5. One should compare 2) with [6], Theorem 6.8 (where analogous statement for Deligne-Lusztig representations is proved). In fact, one can show also the following generalization of (4.6):

$$\dim \mathrm{Hom}_{G(\mathbb{F}_q)}(V_{calL_1,w_1}, V_{\mathcal{L}_2,w_2}) = \#\{w \in W |\ w(\mathcal{L}_2) = \mathcal{L}_1, \text{ and } w_1^{-1}\mathrm{Fr}(w)w_2 = w\} \tag{4.7}$$

We will sketch the proof of (4.7) in the next section.

**Example.** Let $w = 1$ and suppose for simplicity that $\mathcal{L}$ is quasi-regular. Let $\theta = \mathrm{tr}(\mathcal{L})$ be the corresponding character of $T(\mathbb{F}_q)$. Then it is easy to see from 4.3.1 that in this case we have

$$V_{\mathcal{L}} = \{\phi : X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell |\ f(xt) = \theta(t)f(x) \text{ for any } x \in X(\mathbb{F}_q), t \in T(\mathbb{F}_q)\} \tag{4.8}$$

i.e. $V_{\mathcal{L}}$ is just the corresponding principal series representation.

## 5. Proof of Theorem 6

This section is devoted to the proof of Theorem 6, as well as to some generalization of part 2 of this theorem.

**5.1. Whittaker sheaves.** This subsection is devoted to the proof that $V_{\mathcal{L},w}$ is non-zero. Moreover, we will give an estimate of $\dim \operatorname{Hom}_{G(\mathbb{F}_q)}(V_{\mathcal{L},w}, V_{\mathcal{L},w})$ from below.

5.1.1. *Whittaker category.* Let us choose an $\mathbb{F}_q$-rational maximal unipotent subgroup $U$ in $G$. For any simple root $\alpha$ of $G$ let us denote by $U_\alpha$ the corresponding one-parameter subgroup of $U$.

**Definition 6.** *A character (homomorphism of algebraic groups over $\mathbb{F}_q$) $\varepsilon :\ U \to \mathbb{G}_a$ is called non-degenerate if $\varepsilon|_{U_\alpha}$ is non-trivial for every simple root $\alpha$ of $G$ (here $U_\alpha$ denotes the one-parametric subgroup of $U$, corresponding to $\alpha$).*

Fix a non-degenerate character $\varepsilon$ of $U$. Let $\mathcal{L}(\psi, \varepsilon)$ denote the one-dimensional local system $\varepsilon^*\mathcal{L}_\psi$ on $U$ (recall that in Section 2 we have fixed a non-trivial additive character $\psi$ of $\mathbb{F}_q$ and that we denote by $\mathcal{L}_\psi$ the corresponding Artin-Schreier sheaf). Let also $\mathcal{D}_{\varepsilon,\psi}(X)$ denote the derived category of $(U, \mathcal{L}(\varepsilon, \psi))$-equivariant sheaves on $X$. The functors $\Phi_w$ clearly extend to $\mathcal{D}_{\varepsilon,\psi}(X)$.

**Proposition 8.** *1) The image of $\mathcal{D}_{\varepsilon,\psi}(X)$ under the forgetful functor to $\mathcal{D}(X)$ lies in $\mathcal{D}^0(X)$.*

*2)The category $\mathcal{D}_{\varepsilon,\psi}(X)$ is equivalent to the category $\mathcal{D}(T)$. Moreover, for any choice of the symplectic forms $\omega_i$ (cf. Section 2) there exists a choice of $\varepsilon$ and the above equivalnece of categories, in such a way that under this equivalence the functors $\Phi_w$ will tranform into the geometric action of $W$ on $\mathcal{D}(T)$. Moreover, this equivalence commutes with the functor Fr$^*$.*

This statement is proven in [10] on the level of functions. Here one should just repeat word-by-word the arguments of [10].

5.1.2. *Whittaker model of Kazhdan-Laumon representations.* In what follows we fix $\varepsilon$ and an equivalence $\mathcal{D}_{\varepsilon,\psi}(X) \simeq \mathcal{D}(T)$ which satisfy the conditions of Proposition 8. Let $B$ denote the Borel subgroup, ccontaining $U$. Let $j: \ C \to X$ denote the embedding of the unique open $B$-orbit on $X$. $C$ has a natural action of $B \times T$ and it can be $U \times T$-equivariantly identified with $U \times T$. Choose now $\mathcal{L} \in \mathcal{S}(T)$ and set $W_{\mathcal{L},\varepsilon,\psi} = j_!(\mathcal{L}(\varepsilon,\psi) \boxtimes \mathcal{L})$. The following lemma is straighforward.

**Lemma 7.** *$W_{\mathcal{L},\varepsilon,\psi}$ is the unique (up to an isomorphism) irreducible $(U \times T, \mathcal{L}(\varepsilon,\psi) \boxtimes \mathcal{L})$- equivariant perverse sheaf on $X$.*

It follows from Proposition 8 that if $\mathcal{L} \in \mathcal{S}(T)^{\mathrm{Fr}_w}$ then there is a natural isomorphism

$$\alpha_{\varepsilon,\psi,\mathcal{L}}: \ W_{\mathcal{L},\varepsilon,\psi} \tilde{\to} \mathrm{Fr}^* \Phi_w(W_{\mathcal{L},\varepsilon,\psi}) \tag{5.1}$$

and also $(W_{\mathcal{L},\varepsilon,\psi}, \alpha_{\varepsilon,\psi,\mathcal{L}}) \in \mathrm{Perv}^0_{\mathcal{L},w}$. We claim now that the image of $(W_{\mathcal{L},\varepsilon,\psi}, \alpha_{\varepsilon,\psi})$ in $V_{\mathcal{L},w}$ is non-zero. Indeed, it is easy to see that the canonical pairing $\langle , \rangle$ of $W_{\mathcal{L},\varepsilon,\psi}$ with its dual is equal to $1 \neq 0$ which proves what we want.

Thus $V_{\mathcal{L},w} \neq 0$. Working more accurately (varying $\varepsilon$) we can show also that

$$\dim \mathrm{Hom}_{G(\mathbb{F}_q)}(V_{\mathcal{L},w}, V_{\mathcal{L},w}) \geq \# W_{\mathcal{L}}^{\mathrm{Fr}_w} \tag{5.2}$$

**5.2. Proof of theorem 6(1).** In this subsection we fix $\mathcal{L}$ and $w$ and we will write $V_{\mathcal{L}}$ instead of $V_{\mathcal{L},w}$. Let us show that $V_{\mathcal{L}}$ is finite-dimensional for any $\mathcal{L} \in \mathcal{S}(T)$. Since we have a canonical perfect $G(\mathbb{F}_q)$-invariant pairing $\langle \cdot, \cdot \rangle$ between $V_{\mathcal{L}}$ and $V_{\mathcal{L}^{-1}}$, it is enough to show that

$$\dim (V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}})^{G(\mathbb{F}_q)} \leq \# W_{\mathcal{L}} \tag{5.3}$$

This is done in the following way.

**Lemma 8.** *One can identify canonically the $G(\mathbb{F}_q) \times G(\mathbb{F}_q)$-module $V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}}$ with a subquotient of $V_{\mathcal{L}^{-1} \boxtimes \mathcal{L}}$. Here $V_{\mathcal{L}^{-1} \boxtimes \mathcal{L}}$ is the representation of $G(\mathbb{F}_q) \times G(\mathbb{F}_q)$ constructed as in the previous section, using the local system $\mathcal{L}^{-1} \boxtimes \mathcal{L}$ on $T \times T$ (note that the basic affine space of $G \times G$ is $X \times X$).*

*Proof.* First of all, there is a natural map $K_{\mathcal{L}^{-1},w} \otimes K_{\mathcal{L},w} \to K_{\mathcal{L}^{-1} \boxtimes \mathcal{L}}$ constructed as $(A, \alpha) \otimes (B, \beta) \mapsto (A \boxtimes B, \alpha \boxtimes \beta)$. It follows easily from the Küneth formula that this

map preserves the natural pairings on $K_{\mathcal{L}^{-1},w}\otimes K_{\mathcal{L},w}$ and $K_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$. Hence the kernel of the composite map $K_{\mathcal{L}^{-1},w}\otimes K_{\mathcal{L},w}\to V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ lies in $K^{\mathrm{null}}_{\mathcal{L}^{-1}}\otimes K_{\mathcal{L}}+K_{\mathcal{L}^{-1}}\otimes K^{\mathrm{null}}_{\mathcal{L}}$. Hence, $V_{\mathcal{L}^{-1}}\otimes V_{\mathcal{L}}$ gets identified with a subquotient of $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$.

□

The lemma shows that it is enough to show that $\dim V^{G(\mathbb{F}_q)}_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}\leq \# W^{\mathrm{Fr}}_{\mathcal{L}}$.

**Lemma 9.** *Any element of $V^{G(\mathbb{F}_q)}_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ can be represented by a $G$-equivariant pair $(A,\alpha)$.*

*Proof.* Indeed, let $v\in V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ be represented by some $(\overline{A},\overline{\alpha})\in\mathcal{D}^0_{\mathcal{L},w}(X)$. For any $G$-variety $Y$ let $\mathcal{D}_G(Y)$ denote the derived category of equivariant constructible $\overline{\mathbb{Q}}_\ell$-sheaves on $Y$ (cf. [2]). Let also $\mathbf{Av}_G:\ \mathcal{D}(Y)\to\mathcal{D}_G(Y)$ denote the functor of "averaging" over $G$. By definition, $\mathbf{Av}_G(A)=p_!a^*A$ where the maps $a$ and $p$ are defined by the following commutative diagram:

$$\begin{tikzcd} & G\times Y \arrow[dl, "a"'] \arrow[dr, "p"] & \\ Y & & Y \end{tikzcd}$$

Here $p$ is the projection on the second variable and $a$ is given by $a((y,g))=g^{-1}y$. The functor $\mathbf{Av}_G[2\dim G]$ is left adjoint to the forgetful functor $\mathcal{D}_G(Y)\to\mathcal{D}(Y)$.

Let us now go back to the case when $Y=X\times X$. We set now

$$(A,\alpha)=(\mathbf{Av}_G(\overline{A}),\ \frac{\mathbf{Av}_G(\overline{\alpha})}{\#G(\mathbb{F}_q)}) \tag{5.4}$$

(note that $\mathbf{Av}_G(\overline{\alpha})$ is defined, since $\Phi_{w,w}$ commutes with the $G$-action).

It follows now easily from the usual Grothendieck-Lefschetz trace formula (for the Frobenius morphism) that the image of $(A,\alpha)$ in $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ is equal to that of $(\overline{A},\overline{\alpha})$, which is $v$. On the other hand, by definition, the pair $(A,\alpha)$ is $G$-equivariant. This finishes the proof. □

5.2.1. *End of the proof.* We claim now that Proposition 6 together with Corollary 4 imply our statement. Indeed, let $y\in W^{\mathrm{Fr}_w}_{\mathcal{L}}$. Then the sheaf $\mathcal{A}^y_{\mathcal{L}}$ defines us a $G$-equivariant object in $\mathrm{Perv}^0_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}(X\times X)$ (by Corollary 4). Hence it defines an element $a^y_{\mathcal{L},w}$ in $(V_{\mathcal{L},w}\otimes V_{\mathcal{L}^{-1},w})^{G(\mathbb{F}_q)}$ and it follows from Proposition 6 and from

Lemma 9 that the elments $a^y_{\mathcal{L},w}$ (for all $y \in W_{\mathcal{L}}$) span $(V_{\mathcal{L},w} \otimes V_{\mathcal{L}^{-1},w})^{G(\mathbb{F}_q)}$. Therefore we see that

$$\dim V^{G(\mathbb{F}_q)}_{\mathcal{L}^{-1}\boxtimes\mathcal{L}} \leq \# W^{\mathrm{Fr}_w}_{\mathcal{L}} \tag{5.5}$$

which by what is explained above implies that $V_{\mathcal{L}}$ is finite-dimensional and that

$$\dim \mathrm{Hom}(V_{\mathcal{L}}, V_{\mathcal{L}}) = \# W^{\mathrm{Fr}_w}_{\mathcal{L}} \tag{5.6}$$

(the last equation follows from (5.5) and from (5.2)).

We claim now that a little more is true. Namely, we claim that the subquotient which appears in the formulation of Lemma 8 coincides with the whole of $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$. Since both $V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}}$ and $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ are finite-dimensional representations of the group $G(\mathbb{F}_q) \times G(\mathbb{F}_q)$ (the fact that $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ is finite-dimensional follows from the same arguments as above, applied to the group $G \times G$), it is enough to show that

$$\dim \mathrm{Hom}_{G(\mathbb{F}_q)\times G(\mathbb{F}_q)}(V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}}, V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}}) = \dim \mathrm{Hom}_{G(\mathbb{F}_q)\times G(\mathbb{F}_q)}(V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}, V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}) \tag{5.7}$$

However the left hand side of (5.7) is equal to $(\# W^{\mathrm{Fr}_w}_{\mathcal{L}})^2$ by (5.6) and the right hand side of (5.7) is equal to $(\# W^{\mathrm{Fr}_w}_{\mathcal{L}})^2$ again by (5.6), but applied to the group $G \times G$ and the $T \times T$ local system $\mathcal{L} \boxtimes \mathcal{L}^{-1}$. Hence (5.7) holds, which finishes the proof.

We will denote by $\theta : V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}} \tilde{\to} V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ the resulting isomorphism.

**5.3. The inverse map.** We now want to construct an inverse map $\phi : V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}} \to V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}}$.

Let $\mathcal{K} \in \mathcal{D}_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}(X \times X)$. Then we define a functor $\Phi_{\mathcal{K}} : \mathcal{D}_{\mathcal{L}} \to \mathcal{D}_{\mathcal{L}}$ by putting

$$\Phi_{\mathcal{K}}(A) = p_{2!}(p_1^* A \otimes \mathcal{K}) \tag{5.8}$$

where $A \in \mathcal{D}_{\mathcal{L}}(X)$ and $p_1, p_2 : X \times X \to X$ are the natural projections.

Since we have assumed that $\mathcal{K} \in \mathcal{D}_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}(X \times X)$ it follows that we are given an isomorphism of functors

$$\Phi_{\mathcal{K}} \simeq (\mathrm{Fr}^* \circ \Phi_w) \circ \Phi_{\mathcal{K}} \circ (\mathrm{Fr}^* \circ \Phi_w) \tag{5.9}$$

Hence if we are given $A \in \mathcal{D}_{\mathcal{L},w}(X)$, i.e. $A$ is endowed with an isomorphism $\mathrm{Fr}^*\Phi_w(A) \simeq A$ then using (5.9) we may also identify $\Phi_{\mathcal{K}}(A)$ with $\mathrm{Fr}^*\Phi_w(\Phi_{\mathcal{K}}(A))$. Thus $\Phi_{\mathcal{K}}$ induces a map from $K_{\mathcal{L},w}$ to $K_{\mathcal{L},w}$ which we will denote by $\phi_{\mathcal{K}}$.

**Proposition 9.** *Let $\mathcal{K} \in \mathcal{D}_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}(X \times X)$. Then*

(1) $\phi_{\mathcal{K}}(K^{null}_{\mathcal{L},w}) \subset K^{null}_{\mathcal{L},w}$

(2) *Suppose that the image of $\mathcal{K}$ in $K_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}$ lies in $K_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}$. Then*

$$\phi_{\mathcal{K}}(K_{\mathcal{L},w}) \subset K^{null}_{\mathcal{L},w} \tag{5.10}$$

Proposition 9 implies that the assignement $\mathcal{K} \to \phi_{\mathcal{K}}$ descends to a well defined map $\phi:\ V_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)} \to \mathrm{End}\ V_{\mathcal{L},w} = V_{\mathcal{L}^{-1},w} \otimes V_{\mathcal{L},w}$.

*Proof.* Let us prove (2). The proof of (1) is analogous.

So, suppose that we are given $\mathcal{K}$ as above whose image in $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ vanishes. In particular, $\mathcal{K}$ is endowed with an isomorphism $\gamma : \mathcal{K} \tilde{\to} \mathrm{Fr}^*\Phi_{w,w}(\mathcal{K})$.

Let now $(A,\alpha),(B,\beta) \in \mathcal{D}_{\mathcal{L},w}(X)$. Then by (5.9) the complex $\Phi_{\mathcal{K}}(A)$ is also endowed with an isomorphism $\Phi_{\mathcal{K}}(A) \simeq \mathrm{Fr}^*\Phi_w(\Phi_{\mathcal{K}}(A)$ which we will denote by $\Phi_{\mathcal{K}}(\alpha)$. Thus we may consider the endomorphism $\phi(\Phi_{\mathcal{K}}(\alpha),\beta)$ of $\mathrm{RHom}(\Phi_K(A),B)$. We must show that $\mathrm{Tr}(\phi(\Phi_{\mathcal{K}}(\alpha),\beta)) = 0$.

On the other hand, one has

$$\mathrm{RHom}(\Phi_{\mathcal{K}}(A),B) = \mathrm{RHom}(p_{2!}(p_1^*A \otimes \mathcal{K}),B) = \\ \mathrm{RHom}(p_1^*A \otimes \mathcal{K}, p_2^!B) = \mathrm{RHom}(\mathcal{K}, DA \boxtimes B)$$

It is easy to see that under this identifications the map $\phi(\Phi_{\mathcal{K}}(\alpha),\beta)$ goes to $\phi(\gamma,\alpha\boxtimes\beta)$ (the latter is an endomorphism of $\mathrm{RHom}(\mathcal{K}, DA \boxtimes B)$.

Now, the fact that $\mathcal{K}$ vanishes in $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}}$ implies that $\mathrm{Tr}(\phi(\gamma,\alpha\boxtimes\beta)) = 0$. Hence $\mathrm{Tr}(\phi(\Phi_{\mathcal{K}}(\alpha),\beta)) = 0$, which finishes the proof. $\square$

**Lemma 10.** (1) $\phi \circ \theta = \#T_w(\mathbb{F}_q)\,\mathrm{id}$

(2) *Let $m(\mathcal{K})$ denote the matrix coefficient sheaf, corresponding to $\mathcal{K}$ (cf.* 3.2*) with the Weil structure, defined as in* 3.6*. Then the $\overline{\mathbb{Q}}_\ell$-valued function $\frac{\mathrm{tr}(m(\mathcal{K}))}{\#T_w(\mathbb{F}_q)}$ is equal to the matrix coefficient of the image of $\mathcal{K}$ in $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}} = V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}}$.*

The lemma is proved by a direct computation, which is left to the reader.

**5.4. A variant.** Let us again assume that we are given $\mathcal{K} \in \mathcal{D}_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}$. Then, analogously to the definition of modified matrix coefficients, we may define a slightly different version of the functor $\Phi_{\mathcal{K}}$ which we will denote by $\widetilde{\Phi}_{\mathcal{K}}$. Namely, suppose that we are given $A \in \mathcal{D}_{\mathcal{L}}(X)$. Then there exists some $\mathcal{F} \in \mathcal{D}(\mathcal{B} \times X)$ whose pull-back to $X \times X$ can be identified with $p_1^* A \otimes \mathcal{K}$. Thus we define

$$\widetilde{\Phi}_{\mathcal{K}}(A) = \widetilde{p}_{2!}\mathcal{F} \tag{5.11}$$

where $\widetilde{p}_2 : \mathcal{B} \times X \to X$ is the projection to the second multiple. It is again easy to see that the assignment $\mathcal{K} \to \widetilde{\Phi}_{\mathcal{K}}$ reduces to a well-defined morphism $\mathcal{K} \to \widetilde{\phi}_{\mathcal{K}}$ from $K_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}$ to $\mathrm{Hom}(K_{\mathcal{L},w}, K_{\mathcal{L},w})$.

**Proposition 10.** *Let $\mathcal{K} \in \mathcal{D}_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}(X \times X)$. Then*

1. $\widetilde{\phi}_{\mathcal{K}}(K^{null}_{\mathcal{L},w}) \subset K^{null}_{\mathcal{L},w}$
2. *Suppose that the image of $\mathcal{K}$ in $K_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}$ lies in $K_{\mathcal{L}^{-1}\boxtimes\mathcal{L},(w,w)}$. Then*

   $$\widetilde{\phi}_{\mathcal{K}}(K_{\mathcal{L},w}) \subset K^{null}_{\mathcal{L},w} \tag{5.12}$$

   *It follows from the above that the correspondence $\mathcal{K} \to \widetilde{\phi}_{\mathcal{K}}$ defines a map $\widetilde{\phi} : V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}} \to \mathrm{End}\,(V_{\mathcal{L}}$.*
3. $\widetilde{\phi} \circ \psi = \mathrm{id}$
4. *Let $\widetilde{m}(\mathcal{K})$ denote the modified matrix coefficient sheaf, corresponding to $\mathcal{K}$ (cf. 3.3) with the Weil structure, defined as in 3.6. Then the $\overline{\mathbb{Q}}_\ell$-valued function $\mathrm{tr}(\widetilde{m}(\mathcal{K}))$ is equal to the matrix coefficient of the image of $\mathcal{K}$ in $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}} = V_{\mathcal{L}^{-1}} \otimes V_{\mathcal{L}}$.*

The proof is analogous to the proof of Proposition 9.

**5.5. Proof of Theorem 6**(2)**.** In order to show that the character of $V_{\mathcal{L}}$ is equal to $\mathrm{tr}(\mathcal{K}_{\mathcal{L}})$, we must show that the image of $\mathcal{A}_{\mathcal{L}}$ in $\mathrm{Hom}(V_{\mathcal{L}}, V_{\mathcal{L}})$ is equal to the identity element. This follows easily from the following result.

**Lemma 11.** *The functor $\widetilde{\Phi}_{\mathcal{A}_{\mathcal{L}}}$, defined by the sheaf $\mathcal{A}_{\mathcal{L}}$, is canonically isomorphic to identity functor. This isomorphism of functors commutes with the functor $\Phi_w \circ Fr^*$.*

The lemma is straightforward. On the other hand, together with 10 it easily implies that the character of $V_{\mathcal{L}}$ is equal to $\mathrm{tr}(\mathcal{K}_{\mathcal{L}})$, since it implies that $\mathcal{A}_{\mathcal{L}}$ represents the identity element in $V_{\mathcal{L}^{-1}\boxtimes\mathcal{L}} = \mathrm{Hom}(V_{\mathcal{L}}, V_{\mathcal{L}})$.

## 6. Trace formula

Throughout this section $X$ is a scheme of finite type over $k$, where $k$ is an algebraically closed field. For such a scheme we denote by $\mathcal{D}(X)$ the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves (in étale topology) (see e.g. [4]). In this section we suggest a version of Lefschetz-Verdier trace formula in which the role of a geometric correspondence is played by an object of $\mathcal{D}(X \times X)$.

**6.1. Kernels and functors.** Recall that Verdier duality functor is an equivalence of triangulated categories $D : \mathcal{D}(X)^{op} \to \mathcal{D}(X) : A \mapsto R\underline{\mathrm{Hom}}(A, D_X)$ (where $D_X = p^!\overline{\mathbb{Q}}_\ell$, $p$ is the projection to $\mathrm{Spec}(k)$), such that $D^2 \simeq \mathrm{Id}$,

For any $A, B \in \mathcal{D}(X)$ there is a natural isomorphism

$$D(R\underline{\mathrm{Hom}}(A, B)) \simeq A \otimes DB. \tag{6.1}$$

We consider functors $\Phi_K = \Phi_{K,!} : \mathcal{D}(X) \to \mathcal{D}(X)$ associated with kernels $K \in \mathcal{D}(X \times X)$:

$$\Phi_K(A) = p_{2!}(p_1^* A \otimes K). \tag{6.2}$$

Another kind of functors from $\mathcal{D}(X)$ to itself is obtained when composing $\Phi_K$ with Verdier duality. Namely, for $K \in \mathcal{D}(X \times X)$ there is a functor

$$\Psi_K(A) = p_{2*}(R\underline{\mathrm{Hom}}(K, p_1^! A)) \tag{6.3}$$

and the canonical isomorphism of functors

$$\Psi_K \simeq D \circ \Phi_K \circ D, \tag{6.4}$$

derived from (6.1) and standard isomorphisms $D \circ p_{2!} \simeq p_{2*} \circ D$, $D \circ p_1^* \simeq p_1^! \circ D$.

Let us introduce also the relative duality functor $D_{p_1} : \mathcal{D}(X \times X) \to \mathcal{D}(X \times X)$ by setting $D_{p_1}(K) = R\underline{\mathrm{Hom}}(K, p_1^! \overline{\mathbb{Q}}_{l,X})$. Note that we have a canonical morphism of functors $\mathrm{Id} \to D_{p_1}^2$ which, however, is not an isomorphism in general. Now we claim that there is a canonical morphism of functors

$$\Phi_K \to \Psi_{D_{p_1}(K)} \tag{6.5}$$

which is constructed as the following composition

$$\Phi_K(A) = p_{2!}(K \otimes p_1^* A) \to$$
$$p_{2!}(D_{p_1}^2(K) \otimes p_1^* A) \widetilde{\to} p_{2!}(R\underline{\mathrm{Hom}}(D_{p_1}(K), p_1^! \overline{\mathbb{Q}}_{l,X}) \otimes p_1^* A) \to$$
$$\to p_{2!}(R\underline{\mathrm{Hom}}(D_{p_1}(K), p_1^! A)) = \Psi_{D_{p_1}(K)}(A)$$

here we used the canonical morphism $p_1^! \overline{\mathbb{Q}}_{l,X} \otimes p_1^* A \to p_1^! A$ which corresponds to the natural morphism $p_1^* A \to R\underline{\mathrm{Hom}}(p_1^! \overline{\mathbb{Q}}_{l,X}, p_1^! A)$.

**6.2. Trace functions.** With every $K \in \mathcal{D}(X \times X)$ we associate a finite-dimensional vector space $V_K = \mathrm{Hom}(K, \Delta_* D_X)$ where $\Delta : X \to X \times X$ is the diagonal embedding. Then there is a canonical isomorphism

$$V_K \simeq H_c^0(X, \Delta^* K)^*, \tag{6.6}$$

which is constructed as follows:

$$V_K = \mathrm{Hom}(K, \Delta_* D_X) \simeq \mathrm{Hom}(\Delta^* K, D_X) \simeq H^0(X, D\Delta^* K) \simeq H_c^0(X, \Delta^* K)^*$$

where the last isomorphism is given by Verdier duality.

Let $\Phi_K : \mathcal{D}(X) \to \mathcal{D}(X)$ be a functor defined by (6.2), $A$ be an object of $\mathcal{D}(X)$ equipped with a morphism $\alpha : \Phi_K A \to A$. Then we can associate with $\alpha$ an element $t_{K,A}(\alpha) \in V_K$ ("trace function") as follows. By adjunction $\alpha$ corresponds to a morphism $K \otimes p_1^* A \to p_2^! A$, or equivalently to a morphism $\alpha' : K \to R\underline{\mathrm{Hom}}(p_1^* A, p_2^! A)$. Recall that for every $\mathcal{F}, \mathcal{G} \in \mathcal{D}(X)$ there is a canonical isomorphism

$$R\underline{\mathrm{Hom}}(p_1^* \mathcal{F}, p_2^! \mathcal{G}) \simeq D\mathcal{F} \boxtimes \mathcal{G} \tag{6.7}$$

established in [21] (3.2). This isomorphism can be obtained from the isomorphism $p_2^! \mathcal{G} \simeq D_X \boxtimes \mathcal{G}$ (which is the particular case of (6.7) for $\mathcal{F} = \overline{\mathbb{Q}}_{l,X}$) using the commutation of $R\underline{\mathrm{Hom}}$ with the exterior tensor product. Now we apply (6.7) for $\mathcal{F} = \mathcal{G} = A$ and take the composition of $\alpha'$ with the natural morphism $DA \boxtimes A \to \Delta_*(DA \otimes A) \to \Delta_* D_X$ to get an element $t_{K,A}(\alpha) \in \mathrm{Hom}(K, \Delta_* D_X) = V_K$.

By duality, if we have an object $B \in \mathcal{D}(X)$ equipped with a morphism $\beta : B \to \Psi_K(B)$ then we obtain a morphism $\beta' : \Phi_K(DB) \simeq D\Psi_K B \to DB$, hence, the above construction gives an element $t_{K,DB}(\beta') \in V_K$. Now if we are given an object $A \in \mathcal{D}(X)$ equipped with a morphism $\alpha : A \to \Phi_K(A)$ we can take the composition of

$\alpha$ with the morphism (6.5) $\Phi_K(A) \to \Psi_{D_{p_1}(K)}(A)$ to get a morphism $A \to \Psi_{D_{p_1}}(A)$. Applying the above remark we get an element of $V_{D_{p_1}(K)}$ which we denote by $s_{K,A}(\alpha)$.

Note that we have a canonical isomorphism $p_1^! \overline{\mathbb{Q}}_{l,X} \simeq p_2^* D_X$. Hence, we get a canonical morphism

$$
\begin{aligned}
d_K : \Delta^* D_{p_1} K \widetilde{\to} \Delta^* R\underline{\mathrm{Hom}}(K, p_2^* D_X) &\to R\underline{\mathrm{Hom}}(\Delta^* K, \Delta^* p_2^* D_X) \\
&\simeq R\underline{\mathrm{Hom}}(\Delta^* K, D_X) = D\Delta^* K.
\end{aligned}
\tag{6.8}
$$

**Definition 7.** *An object $K \in \mathcal{D}(X \times X)$ is called admissible if the composition of natural maps*

$$d_{K,*} : H_c^0(X, \Delta^* D_{p_1} K) \to H_c^0(X, D\Delta^* K) \to H^0(X, D\Delta^* K)$$

*where the first arrow is induced by $d_K$, is an isomorphism. An object $K$ is called strictly admissible if $d_K$ is an isomorphism and the natural arrow $H_c^0(X, D\Delta^* K) \to H^0(X, D\Delta^* K)$ is an isomorphism.*

**Examples.** 1. If $f : B \to X \times X$ is a correspondence with isolated fixed points, such that $p_1 \circ f$ is étale, then $K = f_*(L)$ is strictly admissible for any local system $L$ on $B$.
2. If $X$ is proper and $d_K$ is an isomorphism then $K$ is strictly admissible. For example, $d_K$ is an isomorphism for $K = L \otimes p_2^* K'$ where $L$ is a local system on $X^2$, $K' \in \mathcal{D}(X)$.

If $K$ is admissible then $d_{K,*}^{-1}$ gives an isomorphism

$$V_K \simeq H^0(X, D\Delta^* K) \widetilde{\to} H_c^0(X, \Delta^* D_{p_1} K)$$

Hence, we obtain a natural pairing

$$\langle \cdot, \cdot \rangle : V_K \otimes V_{D_{p_1} K} \widetilde{\to} H_c^0(X, \Delta^* D_{p_1} K) \otimes \mathrm{Hom}(\Delta^* D_{p_1} K, D_X) \to H_c^0(X, D_X)$$

which induces a perfect pairing

$$\mathrm{Tr}_X \langle \cdot, \cdot \rangle : V_K \otimes V_{D_{p_1} K} \to \overline{\mathbb{Q}}_\ell$$

by composition with the trace map $\mathrm{Tr}_X : H_c^0(X, D_X) \to \overline{\mathbb{Q}}_\ell$.

**Example.** Let $f : B \to X \times X$ be a correspondence with isolated fixed points such that $p_1 \circ f$ is étale, $K = f_*(\overline{\mathbb{Q}}_{l,B})$. Then we have $D_{p_1}(K) \simeq K$. Thus, we have

a (symmetric) non-degenerate form $\chi$ on $V_K \simeq H^0(B\cap\Delta,\overline{\mathbb{Q}}_\ell)^*$, hence on $H^0(B\cap\Delta,\overline{\mathbb{Q}}_\ell)$, where $B\cap\Delta := B\times_{X\times X}\Delta$. The canonical decomposition $H^0(B\cap\Delta,\overline{\mathbb{Q}}_\ell)\simeq \oplus_{x\in B\cap\Delta}\overline{\mathbb{Q}}_\ell e_x$ is orthogonal with respect to this form, and for any $x\in B\cap\Delta$ we have $\chi(e_x,e_x)=m_x$, where $m_x$ is the multiplicity of $x$ in the intersection-product $B\cdot\Delta$. Assume in addition that $p_2\circ f$ is étale. Then the trace function $t_{K,A}(\alpha)$ defined above for an object $A\in\mathcal{D}(X)$ and a morphism $\alpha:\Phi(A)\to A$ is given by $t_{K,A}(\alpha)(e_x)=\mathrm{Tr}(\alpha_x,A_{\bar{x}})$ where $\bar{x}=p_1f(x)=p_2f(x)\in X$, the endomorphism $\alpha_x$ of $A_{\bar{x}}$ is the restriction to $x\in B$ of the morphism $(p_1f)^*A\to(p_2f)^!A\simeq(p_2f)^*A$ corresponding to $\alpha$.

**6.3. Formula.** Let $A,B\in\mathcal{D}(X)$ be a pair of objects equipped with morphisms $\alpha:A\to\Phi_K(A)$ and $\beta:\Phi_K(B)\to B$. Then we have an induced endomorphism of graded vector spaces:

$$\phi(\alpha,\beta): R\operatorname{Hom}(A,B)\to R\operatorname{Hom}(\Phi_K(A),\Phi_K(B))\to R\operatorname{Hom}(A,B). \tag{6.9}$$

Let $\phi_n(\alpha,\beta):\operatorname{Hom}^n(A,B)\to\operatorname{Hom}^n(A,B)$ be the induced morphisms. Then following the usual convention we denote

$$\mathrm{Tr}(\phi(\alpha,\beta))=\sum_i(-1)^i\,\mathrm{Tr}(\phi_i(\alpha,\beta),\operatorname{Hom}^i(A,B)). \tag{6.10}$$

On the other hand, we have defined trace functions $s_{K,A}(\alpha)\in V_{D_{p_1}K}$ and $t_{K,B}(\beta)\in V_K$. Thus, if $K$ is admissible we can define the scalar product $\mathrm{Tr}_X\langle s_{K,A}(\alpha),t_{K,B}(\beta)\rangle\in\overline{\mathbb{Q}}_\ell$.

**Conjecture**. *Assume that $X$ is proper and $K$ is admissible. Then*

$$\mathrm{Tr}_X\langle s_{K,A}(\alpha),t_{K,B}(\beta)\rangle=\mathrm{Tr}(\phi(\alpha,\beta)). \tag{6.11}$$

Below we will prove this under some additional technical assumptions. One of these assumptions which we were unable to check has to do with some functoriality of the construction of the trace map. Namely, for every admissible kernel $L$ and an object $E$ in $\mathcal{D}(X)$ we have defined the trace map

$$s_{L,E}:\operatorname{Hom}(p_2^*E,p_1^*E\otimes L)\simeq\operatorname{Hom}(E,\Phi_L(C))\to V_{D_{p_1(L)}}\simeq H^0(X,\Delta^*L)$$

(the first isomorphism here is due to the fact that $X$ is proper). Now let $f:X'\to X$ be a proper morphism such that $(f\times f)^*L$ is admissible. Then we'll say that the

triple $(f, L, E)$ *behaves functorially* if the following diagram commutes:

$$
\begin{array}{ccc}
\operatorname{Hom}(p_2^*E, p_1^*E \otimes L) & \xrightarrow{s_{L,E}} & H^0(X, \Delta^*L) \\
\downarrow & & \downarrow \\
\operatorname{Hom}(p_2^*f^*E, p_1^*f^*E \otimes (f \times f)^*L) & \xrightarrow{s_{(f\times f)^*L, f^*E}} & H^0(X', f^*\Delta^*L).
\end{array}
\tag{6.12}
$$

**Theorem 7.** *Assume that $X$ is proper, $K$ is admissible, the triple*

$$(\Delta : X \to X \times X, L = p_{13}^*K \otimes p_{24}^*D_{p_1}K, E = A \boxtimes \overline{B})$$

*behaves functorially, and the morphism $\beta$ factorizes as follows*

$$\beta : \Phi_K(B) \to \Psi_{D_{p_1}(K)}(B) \xrightarrow{\widetilde{\beta}} B,$$

*where the first arrow is given by the canonical morphism* (6.5). *Then the formula* (6.11) *holds.*

*Proof.* The idea of the proof is to reduce (6.11) to the usual Lefschetz-Verdier formula. By adjointness and duality the pair of morphisms $\alpha$ and $\beta$ corresponds to a pair of morphisms $\alpha' : p_2^*A \to p_1^*A \otimes K$ and $\beta' : p_2^*\overline{B} \to R\underline{\operatorname{Hom}}(K, p_1^!(\overline{B}))$ where $\overline{B} = D(B)$. Taking the tensor product of $\alpha'$ and $\beta'$ we obtain the morphism

$$p_2^*(A \otimes \overline{B}) \to p_1^*A \otimes K \otimes R\underline{\operatorname{Hom}}(K, p_1^!(\overline{B})) \to p_1^*A \otimes p_1^!\overline{B} \to p_1^!(A \otimes \overline{B}),$$

where the last arrow is induced by the canonical isomorphism $p_1^!E \simeq p_1^*E \otimes p_2^*D_X$ for any $E \in \mathcal{D}(X)$. Let $C = A \otimes \overline{B}$, $\gamma : p_2^*C \to p_1^!C$ be the morphism defined above. Applying Lefschetz-Verdier formula (Thm. 3.3 of [21]) to $\gamma$ and the diagonal correspondence $p_1^*C \to \Delta_*C \to p_2^!C$ we obtain the equality

$$\operatorname{Tr}(\gamma_*) = \operatorname{Tr}_X(\gamma_\Delta) \tag{6.13}$$

where the LHS is the trace of the induced map $\gamma_* : R\Gamma(X, C) \to R\Gamma(X, C)$. The RHS is obtained by applying the the trace map $\operatorname{Tr}_X : H_c^0(X, D_X) \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ to the morphism $\gamma_\Delta : \overline{\mathbb{Q}}_{l,X} \to D_X$ obtained by adjointness from the following morphism induced by $\gamma$:

$$\overline{\mathbb{Q}}_{l,X^2} \to R\underline{\operatorname{Hom}}(p_2^*C, p_1^!C) \simeq C \boxtimes DC \to \Delta_*(C \otimes DC) \to \Delta_*D_X$$

It follows from Lemmas 12 and 13 below that $\operatorname{Tr}(\phi(\alpha, \beta)) = \operatorname{Tr}(\gamma_*)$ and $\operatorname{Tr}_X(\gamma_\Delta) = \operatorname{Tr}_X\langle s_{K,A}(\alpha), t_{K,B}(\beta)\rangle$. Hence, (6.11) follows from (6.13). $\square$

**Lemma 12.** *Let $X$ be proper. Then under the isomorphism*

$$R\operatorname{Hom}(A,B)^* \simeq R\Gamma(X, D(R\underline{\operatorname{Hom}}(A,B))) \simeq R\Gamma(X, A\otimes \overline{B})$$

*one has $\phi(\alpha,\beta)^* = \gamma_*$ where $\phi(\alpha,\beta)^*$ is the dual operator to* (6.9).

*Proof*By definition $\phi = \phi(\alpha,\beta)$ is the composition of the natural maps

$$f : R\operatorname{Hom}(A,B) \to R\operatorname{Hom}(p_1^*A\otimes K, p_1^*B\otimes K)$$

and

$$g : R\operatorname{Hom}(p_1^*A\otimes K, p_1^*B\otimes K) \to R\operatorname{Hom}(\Phi_K(A), \Phi_K(B))$$

and the map

$$h = h(\alpha,\beta) : R\operatorname{Hom}(\Phi_K(A), \Phi_K(B)) \to R\operatorname{Hom}(A,B)$$

induced by $\alpha$ and $\beta$. We have the following natural isomorphisms:

$R\operatorname{Hom}(A,B)^* \simeq R\Gamma(X, A\otimes\overline{B})$
$R\operatorname{Hom}(p_1^*A\otimes K, p_1^*B\otimes K)^* \simeq R\Gamma(X\times X, p_1^*A\otimes K\otimes D(p_1^*B\otimes K)) \simeq$
$\simeq R\Gamma(X\times X, p_1^*A\otimes K\otimes R\underline{\operatorname{Hom}}(K, p_1^!\overline{B}))$,
$R\operatorname{Hom}(\Phi_K(A), \Phi_K(B))^* \simeq R\Gamma(X, \Phi_K(A)\otimes D(\Phi_K(B))) \simeq R\Gamma(X, \Phi_K(A)\otimes\Psi_K(\overline{B}))$.

Under these identifications $f^*$ is induced by the canonical morphism

(6.14)
$$p_{1!}(p_1^*A\otimes K\otimes R\underline{\operatorname{Hom}}(K, p_1^!\overline{B})) \to p_{1!}(p_1^*A\otimes p_1^!\overline{B}) \to p_{1!}p_1^!(A\otimes\overline{B}) \to A\otimes\overline{B},$$

$g^*$ is induced by the morphism

$$\Phi_K(A)\otimes\Psi_K(\overline{B}) \to p_{2^*}(p_1^*A\otimes K)\otimes p_{2*}(R\underline{\operatorname{Hom}}(K, p_1^!\overline{B})) \to$$
$$p_{2*}(p_1^*A\otimes K\otimes R\underline{\operatorname{Hom}}(K, p_1^!\overline{B}))$$

and $h$ is induced by the morphism

$$\alpha\otimes D(\beta) : A\otimes\overline{B} \to \Phi_K(A)\otimes\Psi_K(\overline{B})$$

It follows that $g^*h^*$ is induced by the morphism

$$A\otimes\overline{B} \to p_{2*}p_2^*(A\otimes\overline{B}) \overset{p_{2*}(\alpha'\otimes\beta')}{\to} p_{2*}(p_1^*A\otimes K\otimes R\underline{\operatorname{Hom}}(K, p_1^!\overline{B})) \tag{6.15}$$

and comparing (6.14) and (6.15) with the definition of $\gamma : p_2^*C \to p_1^!C$ (where $C = A \otimes \overline{B}$) we conclude that $\phi^* = f^*g^*h^*$ is equal to the following composition

$$\begin{aligned} R\Gamma(X, C) \to & R\Gamma(X \times X, p_2^*C) \to \\ & R\Gamma(X, p_{1!}p_2^*C) \stackrel{R\Gamma(p_{1!}(\gamma))}{\to} R\Gamma(X, p_{1!}p_1^!C) \to R\Gamma(X, C). \end{aligned}$$

But this is the definition of $\gamma_*$. $\square$

**Lemma 13.** *Under the assumptions of the theorem one has*

$$\gamma_\Delta = \langle s_{K,A}(\alpha), t_{K,B}(\beta)\rangle \tag{6.16}$$

*Proof.* Let us apply the functoriality assumption. We have a morphism $\alpha' : p_2^*A \to p_1^*A \otimes K$ corresponding to $\alpha$ and a morphism $\widetilde{\beta'} : p_2^*\overline{B} \to p_1^*\overline{B} \otimes D_{p_1}K$ corresponding to $\widetilde{\beta}$. Their external product is the morphism

$$\alpha' \boxtimes \widetilde{\beta'} : p_3^*A \otimes p_4^*\overline{B} \to p_1^*A \otimes p_2^*\overline{B} \otimes p_{13}^*K \otimes p_{24}^*D_{p_1}K.$$

Its trace $s_{L,E}(\alpha' \boxtimes \widetilde{\beta'}) \in H^0(X \times X, \Delta^*K \boxtimes \Delta^*D_{p_1}K)$ is equal to the external product of $s_{K,A}(\alpha) \in H^0(X, \Delta^*K)$ and $t_{K,B}(\beta) \in H^0(X, \Delta^*D_{p_1}K)$. Hence, the commutative diagram (6.12) tells us in this case that the element

$$s_{K,A}(\alpha) \cup t_{K,B}(\beta) = \Delta^*s_{L,E}(\alpha' \boxtimes \widetilde{\beta'}) \in H^0(X, \Delta^*(K \otimes D_{p_1}K))$$

is obtained as the trace of the morphism

$$\alpha' \otimes \widetilde{\beta'} : p_2^*(A \otimes \overline{B}) \to p_1^*(A \otimes \overline{B}) \otimes (K \otimes D_{p_1}K).$$

The scalar product $\langle s_{K,A}(\alpha), t_{K,B}(\beta)\rangle$ is the image of $s_{K,A}(\alpha) \cup t_{K,B}(\beta)$ under the map $H^0(X, \Delta^*(K \otimes D_{p_1}K)) \to H^0(X, D_X)$ induced by the natural morphism $K \otimes D_{p_1}K \to p_2^*D_X$. On the other hand, composing $\alpha' \otimes \widetilde{\beta'}$ with the latter morphism we obtain the morphism

$$\gamma : p_2^*(A \otimes \overline{B}) \to p_1^*(A \otimes \overline{B}) \otimes p_2^*D_X$$

introduced above. Hence, its trace $\gamma_\Delta$ is equal to $\langle s_{K,A}(\alpha), t_{K,B}(\beta)\rangle$ as required. $\square$

*Remark* 6. For non-proper $X$ the formula (6.11) holds also for the Frobenius correspondence. One can ask by analogy with Deligne conjecture, whether (6.11) holds for a composition of a given functor with a sufficiently high power of Frobenius correspondence. For example, one can show that it holds for the composition of the symplectic Fourier-Deligne transform with Frobenius correspondence.

## 7. Deligne-Lusztig versus Kazhdan-Laumon representations

In this section we will assume that formula (6.11) holds (in fact, we will make a slightly different assumption, whose proof, however, should be the same as that formula (6.11)). Modulo this assumption we will explain how to connect geometrically the Kazhdan-Laumon representation with Deligne-Lusztig representation ([6]).

**7.1. Deligne-Lusztig representations.** In this subsection we review the definition and basic properties of the Deligne-Lusztig representations (cf. [6]). Our notations, however, will be different from [6].

7.1.1. *The varieties* $X_w$. Recall that in 2.4.2 we have defined for every $w \in W$ certain smooth closed subvariety $Z_w$ inside $X \times X$ of dimension $\dim(X) + l(w)$. Moreover, according to Proposition 2.4.2(4), we know that that $Z_w$ intersects $\Gamma_{\mathrm{Fr}}$ transversally, where $\Gamma_{\mathrm{Fr}}$ denotes the graph of the Frobenius morphism on $X$.

Set now $X_w = Z_w \cap \Gamma_{\mathrm{Fr}}$. Then it follows from the above that $X_w$ is a reduced smooth closed subvariety in $X \times X$, which can be also regarded as a closed subvariety of $X$ using the projection on the first factor. One can easily see that $\dim(X_w) = l(w)$. $X_w$ admits a natural action of the group $G(\mathbb{F}_q) \times T_w(\mathbb{F}_q)$, which is inherited from its action on $X$.

7.1.2. *The representations* $R_{\theta,w}$. Consider now $H^*_c(X_w, \overline{\mathbb{Q}}_\ell)$ – the $\ell$-adic cohomology of $X_w$ with compact supports. Since the finite group $G(\mathbb{F}_q) \times T_w(\mathbb{F}_q)$ acts on $X_w$ it acts also on $H^*_c(X_w, \overline{\mathbb{Q}}_\ell)$.

Thus we may consider $\sum(-1)^i H^i_c(X_w, \overline{\mathbb{Q}}_\ell)$ as a virtual representation of $G(\mathbb{F}_q)$ and decompose it with respect to characters of $T_w(\mathbb{F}_q)$. For any character $\theta : T_w(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}^*$ we will denote by $R_{\theta,w}$ the corresponding virtual representation of $G(\mathbb{F}_q)$. More precisely, for any $i \geq 0$ let

$$H^i_c(X_w, \overline{\mathbb{Q}}_\ell)_\theta = \{\xi \in H^i_c(X_w, \overline{\mathbb{Q}}_\ell) |\ t(\xi) = \theta(t)\xi \quad \text{for any } \xi \in T_w(\mathbb{F}_q)\} \tag{7.1}$$

and we define

$$R_{\theta,w} = \sum_i (-1)^i H^i_c(X_w, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}} \tag{7.2}$$

The next statement is due to P. Deligne and G. Lusztig (cf. [6]) for the case when $q$ is large enough, and it is due to B. Haastert (cf. [8]) in the general case.

**Theorem 8.** *Suppose that $\theta$ is quasi-regular (cf.* 3.1*). Then*

*1) the natural map of forgetting the supports from $H^*_c(X_w, \overline{\mathbb{Q}}_\ell)_\theta$ to $H^*(X_w, \overline{\mathbb{Q}}_\ell)_\theta$ is an isomorphism and $H^i_c(X_w, \overline{\mathbb{Q}}_\ell) = 0$ for $i \neq l(w)$.*

*2) One has*

$$\dim \operatorname{Hom}_{G(\mathbb{F}_q)}(H^{l(w)}_c(X_w, \overline{\mathbb{Q}}_\ell), H^{l(w)}_c(X_w, \overline{\mathbb{Q}}_\ell)) = \# W_\theta^{\mathrm{Fr}_w} \tag{7.3}$$

*In particular, $(-1)^{l(w)} R_{\theta,w}$ is an irreducible representation of $G(\mathbb{F}_q)$ if $\theta$ is regular.*

**7.2. The case of quasi-regular $\mathcal{L}$.** In this subsection we suppose that $\mathcal{L}$ is non-singular. We assume that an analogue of the formula (6.11) holds when $X$ is the basic affine space of $G$, $K = (\mathrm{id} \times \mathrm{Fr}) * \mathbb{Q}_{\ell, Z_w}[l(w)]$, $(A, \alpha) \in \mathrm{Perv}_{\mathcal{L},w}(X)$ and $(B, \beta) \in \mathrm{Perv}_{\mathcal{L},w}(X)$. More precisely, in this case we can define analogues of LHS and RHS of (6.11) as follows. First we notice that $\Phi_K(A) \simeq \Phi_w(\mathrm{Fr}^* A)$ and that the morphism $d_K : \Delta^* D_{p_1} K \to D\Delta^* K$ is an isomorphism in our case. Thus, we can consider the trace function $s_{K,A}(\alpha) \in V_{D_{p_1}K} \simeq H^0(X, \Delta^* K)$. The diagonal action of $T_w(\mathbb{F}_q) = T(\overline{\mathrm{Fr}})^{\mathrm{Fr}_w}$ on $X \times X$ lifts to an action on $K$, such that the induced action of $T_w(\mathbb{F}_q)$ on $H^0(X, \Delta^* K) \simeq H^{l(w)}(X_w, \overline{\mathbb{Q}}_\ell)$ is the natural one. It is easy to see that $s_{K,A}(\alpha)$ lies in $\theta^{-1}$-component of $H^0(X, \Delta^* K)$ where $\theta = \mathrm{tr}\,\mathcal{L}$ is the corresponding character of $T_w(\mathbb{F}_q)$. Similarly, we can consider the trace $t_{K,B}(\beta^{-1})$ of $\beta^{-1} : \Phi_K(B) \to B$ which lies in $H^0(X, \Delta^* D_{p_1} K)_\theta$. Now we have $\Delta^* K \simeq \mathbb{Q}_{\ell, X_w}[l(w)]$, hence, by theorem 8 the natural map $H^0_c(X, \Delta^* K)_\theta \to H^0(X, \Delta^* K)_\theta$ is an isomorphism. In particular, the intersection pairing gives rise to the pairing

$$H^0(X, \Delta^* K)_{\theta^{-1}} \times H^0(X, \Delta^* D_{p_1} K)_\theta \to \overline{\mathbb{Q}}_\ell$$

so we can define $\langle s_{K,A}(\alpha), t_{K,B}(\beta^{-1}) \rangle \in \overline{\mathbb{Q}}_\ell$.

To define the analogue of RHS of (6.11) we remark that in the proper case we used the traces of endomorphisms induced by $\alpha$ and $\beta$ on hypercohomologies of $A \otimes D(B)$. In our case $A \otimes D(B)$ is equivariant with respect to the action of $T$ on $X$, hence it

descends to a sheaf $\overline{A \otimes D(B)}$ on the flag variety $X/T$. Furthermore, since $\alpha$ and $\beta$ were compatible with the action of $T$ we get the induced endomorphisms $\overline{\phi}(\alpha, \beta)$ on hypercohomologies of $\overline{A \otimes D(B)}$ so we can take the alternated sum of their traces. The obtained number differs from the pairing $\langle (A, \alpha), (D(B), D(\beta)^{-1}) \rangle$ defined in 4.3.2 by a constant non-zero multiple.

Next we observe that the morphism of functors $\Phi_K(B) \to \Psi_{D_{p_1}K}(B)$ is an isomorphism, so the last condition of theorem 7 is satisfied. Thus, assuming the functoriality property in the formulation of this theorem we can slightly modify the argument to prove the equality

$$\mathrm{Tr}_X \langle s_{K,A}(\alpha), t_{K,B}(\beta^{-1}) \rangle = \mathrm{Tr}(\overline{\phi}(\alpha, \beta)). \tag{7.4}$$

The following result is obtained as a corollary of the formula (7.4).

**Theorem 9.** *Let $\mathcal{L}$ be a quasi-regular one-dimensional local system on $T$ and let $w \in W$ be an element of the Weyl group, such that $Fr_w^*(\mathcal{L})$ is isomorphic to $\mathcal{L}$. Let $\theta$ be the corresponding character of $T_w(\mathbb{F}_q)$. Then one has canonical isomorphism of $G(\mathbb{F}_q)$-representations*

$$V_{\mathcal{L}} \simeq H_c^{l(w)}(X_w)^*_\theta \simeq R_{\theta,w} \tag{7.5}$$

*Proof.* The proof of the second isomorphism follows merely from Theorem 8. Therefore, it is enough to construct the first isomorphism. This is done in the following way.

We have to define a map $\kappa:\ V_{\mathcal{L}} \to H_c^{l(w)}(X_w)^*_\theta$. We will do it as follows. First, we will define $\overline{\kappa} : K_{\mathcal{L},w} \to H_c^{l(w)}(X_w)^*_\theta$ (which will be a map of $G(\mathbb{F}_q)$-modules) and then show that it is surjective and that its kernel is equal to $K^{\mathrm{null}}_{\mathcal{L},w}$.

So, we define

$$\overline{\kappa}((A, \alpha)) = t_{A,\alpha} \tag{7.6}$$

for any $(A, \alpha) \in \mathrm{Perv}_{\mathcal{L},w}(X)$. It is easy to see that $\overline{\kappa}$ is a well defined map from $K_{\mathcal{L},w}$ to $H_c^{l(w)}(X_w)^*_\theta$, which commutes with $G(\mathbb{F}_q)$-action. We claim, first of all, that $K^{\mathrm{null}}_{\mathcal{L},w}$ contains the kernel of $\overline{\kappa}$. Indeed, suppose that we have some $(A, \alpha) \in \mathrm{Perv}_{\mathcal{L},w}(X)$ whose image in $K_{\mathcal{L},w}$ does not lie in $K^{\mathrm{null}}_{\mathcal{L},w}$. Then there exists some $(B, \beta) \in \mathrm{Perv}_{\mathcal{L}^{-1},w}$, such that $\langle (A, \alpha), (B, \beta) \rangle \neq 0$. It follows from formula (7.4) (which we assumed to hold

in this case), that $\langle t_{A,\alpha}, s_{D(B),D(\beta)^{-1}}\rangle$ is also non-zero. Hence, $\langle t_{A,\alpha}, s_{D(B),D(\beta)^{-1}}\rangle \neq 0$ and therefore $t_{A,\alpha} \neq 0$, which is what we had to prove.

It follows now from the fact that $\operatorname{Ker}\overline{\kappa} \subset K^{\mathrm{null}}_{\mathcal{L},w}$ that the map $\overline{\kappa}$ identifies $V_{\mathcal{L}}$ with a subquotient of $H^{l(w)}_c(X,\overline{\mathbb{Q}}_\ell)^*_\theta$. Therefore, in order to show that $\overline{\kappa}$ in fact descends to an isomorphism between the two, it is enough to show (since $G(\mathbb{F}_q)$ is a finite group) that

$$\dim \operatorname{Hom}_{G(\mathbb{F}_q)}(V_{\mathcal{L}}, V_{\mathcal{L}}) = \dim \operatorname{Hom}_{G(\mathbb{F}_q)}(H^{l(w)}_c(X,\overline{\mathbb{Q}}_\ell)^*_\theta, H^{l(w)}_c(X,\overline{\mathbb{Q}}_\ell)^*_\theta) \tag{7.7}$$

(In fact, we already know, in fact, that the two representations are isomorphic, since the character of both of them is equal to $\operatorname{tr}(\mathcal{K}_{\mathcal{L}})$. However, we want to show that $\overline{\kappa}$ defines an isomorphism between $V_{\mathcal{L}}$ and $(-1)^{l(w)}R_{\theta,w}$ independently of the above computation of their characters).

We know now that the left hand side of (7.7) is equal to $\#W_{\mathcal{L}}^{\mathrm{Fr}_w}$ and the right hand side is equal to $\#W_\theta^{\mathrm{Fr}_w}$. On the other hand, it is easy to see that $W_{\mathcal{L}}^{\mathrm{Fr}_w} = W_\theta^{\mathrm{Fr}_w}$, which finishes the proof. $\square$

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